A functoriality principle for blocks of p-adic linear groups

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Abstract

Bernstein blocks of complex representations of p-adic reductive groups have been computed in a large number of examples, in part thanks to the theory of types à la Bushnell and Kutzko. The output of these purely representation-theoretic computations is that many of these blocks are equivalent. The motto of this paper is that most of these coincidences are explained, and many more can be predicted, by a functoriality principle involving dual groups. We prove a precise statement for groups related to GL_n, and then state conjectural generalizations in two directions: more general reductive groups and/or *integral* l-adic representations.

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1 Main statements

Let F be a p-adic field and let R be a commutative ring in which p is invertible. For $\mathbf{G}$ a reductive group over F, we put $G := \mathbf{G}(F)$ and we denote by $\mathrm{Rep}_R(G)$ the abelian category of smooth representations of G with coefficients in R, and by $\mathrm{Irr}_R(G)$ the set of isomorphism classes of simple objects in $\mathrm{Rep}_R(G)$. We will be mainly interested in the cases $R = \overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{Z}}_\ell$ or $\overline{\mathbb{F}}_\ell$ for ℓ a prime number coprime to p.

Let us assume $R = \overline{\mathbb{Q}}_\ell$ for a while. For a general $\mathbf{G}$, Bernstein [1] has decomposed $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ as a product of indecomposable abelian subcategories called *blocks*. This decomposition is characterized by the property that two irreducible representations belong to the same Bernstein block if and only if their supercuspidal supports are "inertially equivalent".

When $\mathbf{G} = \mathrm{GL}_n$, Bushnell and Kutzko [3] [4] have proved that each block is equivalent to the category of modules over an algebra of the form $\mathcal{H}(n_1, q^{f_1}) \otimes_{\overline{\mathbb{Q}}_\ell} \cdots \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{H}(n_r, q^{f_r})$ where $\mathcal{H}(n, q)$

denotes the extended Iwahori–Hecke algebra of type A_{n-1} and parameter q (the size of the residual field of F). This shows in particular that, up to taking tensor product of categories, all blocks of linear groups look "the same". More precisely, joint with Borel's theorem, their work shows that any Bernstein block is equivalent to the *principal* block of a product of general linear groups. Here, as usual, the "principal block" is the one that contains the trivial representation.

The main result of this paper is a "Langlands-dual" explanation of this redundancy among blocks of linear groups. It will appear as a functoriality principle for blocks, and also will point to particularly nice equivalences, related to the usual functoriality principle for irreducible representations.

However, the main interest of the paper probably lies in the conjectural natural generalizations of the main result. These speculations will involve more general reductive groups $\mathbf{G}$ and/or coefficients $R = \overline{\mathbb{Z}}_\ell$ or $\overline{\mathbb{F}}_\ell$ for ℓ a prime number coprime to p.

1.1 Functoriality for $\overline{\mathbb{Q}}_\ell$-blocks of groups of GL-type

We say that $\mathbf{G}$ is "of GL-type" if it is isomorphic to a product of restriction of scalars of general linear groups over finite extensions of F.

1.1.1 *Langlands parametrization of $\overline{\mathbb{Q}}_\ell$-blocks.* — Let ${}^L\mathbf{G} = \hat{\mathbf{G}} \rtimes W_F$ be "the" dual group of $\mathbf{G}$, where W_F is the Weil group of F. For a general $\mathbf{G}$, Langlands' parametrization conjecture predicts the existence of a finite-to-one map $\pi \mapsto \varphi_\pi$

$$\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G) \longrightarrow \Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell) := \{\varphi : W_F \ltimes \overline{\mathbb{Q}}_\ell \longrightarrow {}^L\mathbf{G}(\overline{\mathbb{Q}}_\ell)\}_{/\sim}$$

where the right hand side denotes the set of "admissible"[1] L-parameters for $\mathbf{G}$ modulo conjugacy by $\hat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$. For $\mathbf{G}$ of GL-type, this parametrization follows from the Langlands correspondence of [8] [11] for GL_n via the non-commutative Shapiro lemma, and it is a *bijection*. Moreover, this correspondence is known to be compatible with parabolic induction in the following sense. If π is an irreducible subquotient of some induced representation $i_{M,P}^G(\sigma)$, then $\varphi_{\pi|W_F} \sim {}^L\iota \circ \varphi_{\sigma|W_F}$, where ${}^L\iota : {}^L\mathbf{M} \hookrightarrow {}^L\mathbf{G}$ is any embedding dual to $\mathbf{M} \hookrightarrow \mathbf{G}$ (note that $\mathbf{M}$ is also of GL-type). As a consequence, for two irreducible representations π, π' in the same Bernstein block, we have $\varphi_{\pi|I_F} \sim \varphi_{\pi'|I_F}$, hence we get a decomposition

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G) = \prod_{\phi \in \Phi_{\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)} \mathrm{Rep}_\phi(G)$$

where the index set $\Phi_{\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell) = H^1(I_F, \hat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell))^{W_F}$ is the set of $\hat{\mathbf{G}}$-conjugacy classes of *admissible inertial parameters*, i.e. continuous[2] sections $I_F \longrightarrow {}^L\mathbf{G}(\overline{\mathbb{Q}}_\ell)$ that admit an extension to an admissible L-parameter in $\Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, and $\mathrm{Rep}_\phi(G)$ consists of all smooth $\overline{\mathbb{Q}}_\ell$-representations of G, any irreducible subquotient π of which satisfies $\varphi_{\pi|I_F} \sim \phi$.

It is "well known" that $\mathrm{Rep}_\phi(G)$ is actually a Bernstein block, see Lemma 2.4.1, so that we get a parametrization of blocks of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ by $\Phi_{\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$. In this parametrization, the principal block corresponds to the trivial parameter $i \mapsto (1, i)$, which may explain why it is sometimes referred to as the "unipotent" block.

[1] "admissible" implies in particular that the image of W_F consists of semi-simple elements, an element of ${}^L\mathbf{G}$ being semi-simple if its projection on any algebraic quotient $\hat{\mathbf{G}} \rtimes \Gamma_{F'/F}$ is semi-simple.

[2] In all this discussion $\hat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$ is equipped with the discrete topology.

1.1.2 *Functorial transfer of $\overline{\mathbb{Q}}_\ell$-blocks.* — Let $\mathbf{G}'$ be another group of GL-type over F, and suppose given a morphism[3] of L-groups $\xi : {}^L\mathbf{G}' \longrightarrow {}^L\mathbf{G}$. The composition map $\varphi' \mapsto \xi \circ \varphi$ on L-parameters translates into a map $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G') \xrightarrow{\xi_*} \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$, known as the "(local) Langlands' transfer induced by ξ". A natural question to ask is whether this Langlands transfer can be extended to non-irreducible representations in a functorial way. In general there seems to be little hope, but the following result shows that it becomes possible under certain circumstances.

Let us fix an admissible inertial parameter $\phi' : I_F \longrightarrow {}^L\mathbf{G}$ and put $\phi := \xi \circ \phi'$. As usual, let $C_{\hat{\mathbf{G}}}(\phi)$ denote the centralizer in $\hat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$ of the image of the inertial parameter ϕ.

Theorem. — Suppose that ξ induces an isomorphism $C_{\hat{\mathbf{G}}'}(\phi') \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$. Then there is an equivalence of categories $\mathrm{Rep}_{\phi'}(G') \xrightarrow{\sim} \mathrm{Rep}_\phi(G)$ that interpolates the Langlands transfer ξ_* on irreducible representations.

Remark.– We expect that such an equivalence will be unique, up to natural transformation. In fact, the equivalences that we will exhibit are also compatible with parabolic induction, and this extra compatibility makes them unique.

1.1.3 Let us give some examples that may shed light on the statement.

i) Suppose that ξ is a dual Levi embedding. This means that $\mathbf{G}'$ may be embedded as a F-Levi subgroup of $\mathbf{G}$. Fix a parabolic subgroup P of G with Levi component G', and let $[M', \pi']_{G'}$ be the inertial equivalence class of supercuspidal pairs associated with ϕ'. Then the condition in the theorem is equivalent to the requirement that the stabilizer of $[M', \pi']_{G'}$ in G is G'. In this situation, it is well-known that the normalized parabolic induction functor i_P provides an equivalence of categories as in the theorem.

ii) Suppose that ξ is a base change homomorphism $\mathrm{GL}_n \times W_F \longrightarrow {}^L(\mathrm{Res}_{F'|F}\mathrm{GL}_n)$, and let ϕ' and ϕ both be trivial. Then the condition in the theorem is met if and only if F' is totally ramified over F. On the other hand, $\mathrm{Rep}_{\phi'}(G')$ is the principal block of $\mathrm{GL}_n(F)$ while $\mathrm{Rep}_\phi(G)$ is the principal block of $\mathrm{GL}_n(F')$. So Borel's theorem tells us they are respectively equivalent to the category of right modules over $\mathcal{H}(n, q_F)$, resp. $\mathcal{H}(n, q_{F'})$. Therefore they are equivalent if and only if F' is totally ramified over F. Moreover, it is not hard to find an equivalence that meets the requirement of the theorem.

iii) Suppose that ξ is an isomorphism of L-groups. Then the conditions of the theorem are met for all ϕ' ! To describe the equivalence, first conjugate under $\hat{\mathbf{G}}$ to put ξ in the form $\hat{\xi} \times \psi$ where $\hat{\xi} : \hat{\mathbf{G}}' \longrightarrow \hat{\mathbf{G}}$ is an épinglage preserving W_F-equivariant isomorphism, and $\psi : W_F \longrightarrow Z(\hat{\mathbf{G}}) \rtimes W_F$. Then $\hat{\xi}$ provides an F-rational isomorphism $\hat{\xi}^\vee : \mathbf{G} \xrightarrow{\sim} \mathbf{G}'$ (well-defined up to conjugacy), and ψ determines a character $\psi^\vee : G \longrightarrow \mathbf{G}_{\mathrm{ab}}(F) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ through local class field theory. The desired equivalence is given by pre-composition under $\xi^\vee$ followed by twisting under $\psi^\vee$. Its compatibility with the Langlands transfer is Proposition 5.2.5 of [7].

iv) Suppose that $\mathbf{G} = \mathrm{GL}_n$ and let ϕ be an inertial parameter of $\mathbf{G}$ that is irreducible as a representation of I_F. Put $\mathbf{G}' = \mathrm{GL}_1$ and $\phi' = 1$ (trivial parameter). Finally let ξ be the product of the central embedding $\mathrm{GL}_1 \hookrightarrow \mathrm{GL}_n$ and any extension $\varphi : W_F \longrightarrow \mathrm{GL}_n$ of ϕ. Then the conditions of the Theorem are met, $\mathrm{Rep}_\phi(G)$ is a cuspidal block and $\mathrm{Rep}_{\phi'}(G)$ is $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(F^\times/\mathcal{O}_F^\times)$. The claimed equivalence is given by $\chi \mapsto \pi \otimes (\chi \circ \det)$ where π corresponds

[3] Since we work with the Weil form of L-groups, we require that a morphism of L-groups carries semi-simple elements to semi-simple elements.

to φ through Langlands' correspondence.

1.1.4 *Reduction to unipotent blocks.* — What makes Theorem 1.1.2 a very flexible statement is that no a priori restriction was made on ξ ; namely its Galois component can be very complicated. In this regard, it is important to work with the Weil form of the L-group. For instance, in the last example above, ϕ' was trivial but all the complexity of the setting was "moved" to the L-homomorphism ξ. This is a simple example of a more general phenomenon that allows to reduce Theorem 1.1.2 to an equivalent statement which deals with a single parameter $\phi \in \Phi_{\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$. We now explain this, and refer to section 2.2 for details.

By definition we may choose an extension φ of ϕ to an L-parameter $W_F \longrightarrow {}^L\mathbf{G}(\overline{\mathbb{Q}}_\ell)$. Conjugation by $\varphi(w)$ in ${}^L\mathbf{G}$ then induces an action of W_F/I_F on $C_{\hat{\mathbf{G}}}(\phi)$ and a factorization :

$$\phi:\ I_F \xrightarrow{i\mapsto(1,i)} C_{\hat{\mathbf{G}}}(\phi) \rtimes W_F \xrightarrow{(z,w)\mapsto z\varphi(w)} {}^L\mathbf{G}(\overline{\mathbb{Q}}_\ell)\ .$$

It turns out that the outer action $W_F \longrightarrow \mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi))$ does not depend on the choice of φ and thus defines a "unique" unramified group $\mathbf{G}_\phi$ over F. Moreover, one checks that this group is of GL-type, and that there exists φ such that all $\varphi(w)$ preserve a given épinglage of $C_{\hat{\mathbf{G}}}(\phi)$. So we get a factorization of the form $\phi : I_F \stackrel{1\times\mathrm{Id}}{\longrightarrow} {}^L\mathbf{G}_\phi \stackrel{\xi_\varphi}{\longrightarrow} {}^L\mathbf{G}$, as considered in Theorem 1.1.2. By construction, the hypothesis of the latter theorem is satisfied, so we get the following

Corollary. — *There is an equivalence of categories* $\mathrm{Rep}_1(G_\phi) \stackrel{\sim}{\longrightarrow} \mathrm{Rep}_\phi(G)$ *that extends the transfer associated to the above* $\xi_\varphi : {}^L\mathbf{G}_\phi \hookrightarrow {}^L\mathbf{G}$.

We like to see this statement as a "moral explanation" to, and a more precise version of, the well known property that any $\overline{\mathbb{Q}}_\ell$-block of a GL_n is equivalent to the principal $\overline{\mathbb{Q}}_\ell$-block of a product of linear groups over extension fields. Also we note that a different choice of extension φ' of ϕ as above will differ from φ by a central unramified cocycle $W_F \longrightarrow Z(\hat{\mathbf{G}})$, and an associated equivalence $\xi_{\varphi'}$ is thus deduced from ξ_φ by twisting by the associated unramified character of G.

Remark.– Again we may ask whether an equivalence as in the Corollary is actually unique. This boils down to the following concrete question. Let $\mathcal{H} = \mathcal{H}(n,q)$ be the extended affine Hecke algebra of type A_n and parameter q. Suppose Φ is an auto-equivalence of categories of $\mathcal{H} - \mathrm{Mod}$ that fixes all simple modules up to equivalence. Is Φ isomorphic to the identity functor ? Even more concretely : if $\mathcal{I}$ is an invertible $\mathcal{H}\otimes_{\overline{\mathbb{Q}}_\ell}\mathcal{H}^{\mathrm{opp}}$-module such that $\mathcal{I}\otimes_{\mathcal{H}} S \simeq S$ for any simple left $\mathcal{H}$-module S, do we have $\mathcal{I} \simeq \mathcal{H}$?

1.1.5 *On the proofs.* — In fact, it is easy to show that Corollary 1.1.4 with (ϕ,φ) allowed to vary, is equivalent to Theorem 1.1.2 with (ϕ',ξ) allowed to vary, see Lemma 2.4.8. Now, to prove the corollary directly, we may first reduce to the case where $\mathbf{G}$ is quasi-simple, *i.e.* of the form $\mathrm{Res}_{F'|F}\mathrm{GL}_n$, then by a Shapiro-like argument to the case $\mathbf{G} = \mathrm{GL}_n$, then, using parabolic induction, to the case where $\mathbf{G}_\phi$ is quasi-simple. In the latter case, $\mathrm{Rep}_\phi(\mathbf{G})$ is cut out by a simple type of [3] and the desired equivalence follows from the computation of Hecke algebras there. Note that the information needed on the simple type is very coarse ; only the degree and residual degree of the field entering the definition of the type is involved here. Details are given in 2.4.

1.1.6 *Variants.* — In the foregoing discussion, we may try to replace admissible parameters with domain I_F by admissible parameters with domain any closed normal subgroup K_F of I_F. The main examples we have in mind are $K_F = P_F$, the wild inertia subgroup, and $K_F = I_F^{(\ell)} := \ker(I_F \longrightarrow \mathbb{Z}_\ell(1))$, which is the maximal closed subgroup of I_F with prime-to-ℓ pro-order. Other possibilities are ramication subgroups of P_F. In any case, an *admissible* K_F*-parameter of* $\mathbf{G}$ will be a morphism $K_F \longrightarrow {}^L\mathbf{G}$ that admits an extension to a usual admissible parameter $W_F \longrightarrow {}^L\mathbf{G}$. By grouping Bernstein blocks, we thus get a product decomposition $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G) = \prod_\phi \mathrm{Rep}_\phi(G)$ where ϕ runs over admissible K_F-parameters up to $\hat{\mathbf{G}}$ conjugacy, and $\mathrm{Rep}_\phi(G)$ is "generated" by the irreducible representations π such that $(\varphi_\pi)_{|K_F} \sim \phi$. For example, in the case $K_F = P_F$, the factor $\mathrm{Rep}_1(G)$ is the level 0 subcategory.

It is then easy to deduce from Theorem 1.1.2 exactly the same statement for K_F-parameters, simply by grouping the equivalences provided by this theorem. In contrast, our arguments in this paper, in particular in paragraph 2.4.6, allow us to prove the natural analogue of Corollary 1.1.4 only when K_F contains P_F.

1.2 Functoriality for $\overline{\mathbb{Z}}_\ell$-blocks of groups of GL-type

For $\mathbf{G}$ of GL-type, Vignéras [15] has obtained a decomposition of $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(G)$ formally analogous to that of Bernstein, where the blocks are indexed by inertial classes of supercuspidal pairs over $\overline{\mathbb{F}}_\ell$. This was further lifted to a decomposition of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$ by Helm in [10]. In general, Vignéras or Helm blocks will *not* be equivalent to categories of modules over an Hecke–Iwahori algebra, and actually even the structure of the principal block of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$ (which may contain non-Iwahori-spherical representations) is not yet well understood.

1.2.1 *Langlands parametrization of* $\overline{\mathbb{Z}}_\ell$*-blocks.* — In exactly the same way as for $\overline{\mathbb{Q}}_\ell$-blocks (see 2.4.2 for some details), Vignéras' Langlands correspondence for $\overline{\mathbb{F}}_\ell$-representations [16] allows one to rewrite the Vignéras–Helm decomposition in the form

$$\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathrm{GL}_n(F)) = \prod_{\bar{\phi}\in\Phi_{\mathrm{inert}}(\mathrm{GL}_n,\overline{\mathbb{F}}_\ell)} \mathrm{Rep}_{\bar{\phi}}(\mathrm{GL}_n(F))$$

with $\Phi_{\mathrm{inert}}(\mathrm{GL}_n, \overline{\mathbb{F}}_\ell)$ the set of equivalence classes of semi-simple n-dimensional $\overline{\mathbb{F}}_\ell$-representations of I_F that extend to W_F. This suggests to use L-groups over $\overline{\mathbb{F}}_\ell$ in order to mimic the transfer of $\overline{\mathbb{Q}}_\ell$-blocks. *However,* in order to get a functoriality property analogous to Theorem 1.1.2, we need to stick to the usual L-groups over $\overline{\mathbb{Q}}_\ell$.

Recall that we have a "semisimplified reduction mod ℓ" map $r_\ell : \Phi_{\mathrm{inert}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \longrightarrow \Phi_{\mathrm{inert}}(\mathrm{GL}_n, \overline{\mathbb{F}}_\ell)$. The basic properties of the Vignéras–Helm decomposition tell us that, denoting by $e_{\bar{\phi}}$ the primitive idempotent in the center $\mathfrak{Z}_{\overline{\mathbb{Z}}_\ell}(G)$ of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$ that cuts out the block $\mathrm{Rep}_{\bar{\phi}}(G)$, we have the equality

$$e_{\bar{\phi}} = \sum_{r_\ell(\phi)=\bar{\phi}} e_\phi, \quad \text{in} \quad \mathfrak{Z}_{\overline{\mathbb{Q}}_\ell}(G).$$

Now recall the "prime-to-ℓ inertia subgroup" $I_F^{(\ell)}$ of 1.1.6, and define $\Phi_{\ell'-\mathrm{inert}}(\mathrm{GL}_n, R)$ as the set of semi-simple n-dimensional R-representations of $I_F^{(\ell)}$ that extend to W_F (here, R is either $\overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$). We have a commutative diagram

$$\begin{array}{ccc} \Phi_{\mathrm{inert}}(\mathrm{GL}_n,\overline{\mathbb{Q}}_\ell) & \xrightarrow{\mathrm{res}} & \Phi_{\ell'-\mathrm{inert}}(\mathrm{GL}_n,\overline{\mathbb{Q}}_\ell) \\ {\scriptstyle r_\ell}\downarrow & & \downarrow{\scriptstyle r_\ell} \\ \Phi_{\mathrm{inert}}(\mathrm{GL}_n,\overline{\mathbb{F}}_\ell) & \xrightarrow[\mathrm{res}]{} & \Phi_{\ell'-\mathrm{inert}}(\mathrm{GL}_n,\overline{\mathbb{F}}_\ell) \end{array}.$$

The reduction map r_ℓ on the right hand side is a bijection because $I_F^{(\ell)}$ has prime-to-ℓ order. Moreover, the restriction map on the bottom is also a bijection because a semisimple $\overline{\mathbb{F}}_\ell$-representation of I_F is determined by its restriction to $I_F^{(\ell)}$ (indeed, it is determined by its Brauer character on ℓ'-elements, but the set of ℓ' elements of I_F is precisely $I_F^{(\ell)}$).

This shows that we may parametrize the Vignéras-Helm blocks by the set $\Phi_{\ell'-\mathrm{inert}}(\mathbf{G},\overline{\mathbb{Q}}_\ell)$, with the restriction map from I_F to $I_F^{(\ell)}$ playing the role of the reduction map r_ℓ. Using the non-commutative Shapiro lemma (see Corollary 2.3.3), we get for any group $\mathbf{G}$ of GL-type a parametrization of blocks of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$ by the set

$$\Phi_{\ell'-\mathrm{inert}}(\mathbf{G},\overline{\mathbb{Q}}_\ell) := \left\{\phi_\ell : I_F^{(\ell)} \longrightarrow {}^L\mathbf{G}(\overline{\mathbb{Q}}_\ell),\ \exists\varphi\in\Phi(\mathbf{G},\overline{\mathbb{Q}}_\ell), \varphi_{|I_F^{(\ell)}} = \phi_\ell\right\}_{/\hat{\mathbf{G}}-\mathrm{conj}}.$$

1.2.2 *Functorial transfer of $\overline{\mathbb{Z}}_\ell$-blocks.* — Now consider again an L-homomorphism ξ : ${}^L\mathbf{G}' \longrightarrow {}^L\mathbf{G}$ of groups of GL-type, fix an admissible parameter $\phi' : I_F^{(\ell)} \longrightarrow {}^L\mathbf{G}'$ and put $\phi := \xi\circ\phi'$. Attached to ϕ is a $\overline{\mathbb{Z}}_\ell$-block $\mathrm{Rep}_{\phi,\overline{\mathbb{Z}}_\ell}(G)$, whose $\overline{\mathbb{Q}}_\ell$-objects form a finite sum of $\overline{\mathbb{Q}}_\ell$-blocks $\mathrm{Rep}_{\phi,\overline{\mathbb{Q}}_\ell}(G) = \prod_{\psi_{|I_F^{(\ell)}}\sim\phi}\mathrm{Rep}_\psi(G)$.

Conjecture. — Suppose that ξ induces an isomorphism $C_{\hat{\mathbf{G}}'}(\phi') \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$, and also that the projection of $\xi(W_F)$ to $\hat{G}(\overline{\mathbb{Q}}_\ell)$ is bounded. Then there is an equivalence of categories $\mathrm{Rep}_{\phi',\overline{\mathbb{Z}}_\ell}(G') \xrightarrow{\sim} \mathrm{Rep}_{\phi,\overline{\mathbb{Z}}_\ell}(G)$ that interpolates the Langlands transfer ξ_* on irreducible $\overline{\mathbb{Q}}_\ell$-representations.

Again we may also conjecture that there is a unique such equivalence of categories, or at least that any equivalence $\mathrm{Rep}_{\phi',\overline{\mathbb{Q}}_\ell}(G') \xrightarrow{\sim} \mathrm{Rep}_{\phi,\overline{\mathbb{Q}}_\ell}(G)$ as in Theorem 1.1.2 extends to an equivalence $\mathrm{Rep}_{\phi',\overline{\mathbb{Z}}_\ell}(G') \xrightarrow{\sim} \mathrm{Rep}_{\phi,\overline{\mathbb{Z}}_\ell}(G)$.

Remark.– Let us take up the base change example 1.1.3 ii). With the notation there, the requirements of the conjecture are met if and only if F' is a totally ramified extension of *degree prime to ℓ.* The conjecture then predicts an equivalence between the principal $\overline{\mathbb{Z}}_\ell$-blocks of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_n(F')$. In fact, it is plausible that such an equivalence exists when F' is *only* assumed to be totally ramified, but in general it won't interpolate the base change of irreducible $\overline{\mathbb{Q}}_\ell$-representations. As an example, put $n=2$, $F=\mathbb{Q}_p$, $\ell|(p+1)$ odd, and $F'=\mathbb{Q}_p(p^{1/\ell})$. In this situation there exists a supercuspidal $\overline{\mathbb{Q}}_\ell$-representation π in the principal $\overline{\mathbb{Z}}_\ell$-block of $\mathrm{GL}_n(F)$ whose base change is a principal series of $\mathrm{GL}_n(F')$. Indeed, take for π the representation that corresponds to the irreducible Weil group representation $\sigma := \mathrm{ind}_{W_{\mathbb{Q}_{p^2}}}^{W_{\mathbb{Q}_p}}(\chi)$ where χ is any character of $W_{\mathbb{Q}_{p^2}} \longrightarrow \overline{\mathbb{Z}}_\ell^\times$ that extends a character $I_{\mathbb{Q}_p} \twoheadrightarrow \mu_\ell \hookrightarrow \overline{\mathbb{Q}}_\ell^\times$.

1.2.3 *Reduction to unipotent $\overline{\mathbb{Z}}_\ell$-blocks.* — Start with an admissible parameter $\phi : I_F^{(\ell)} \longrightarrow {}^L\mathbf{G}$ and choose an extension to some usual parameter $\varphi : W_F \longrightarrow {}^L\mathbf{G}$. The same procedure as in paragraph 1.1.4 provides us with a factorization $\phi : I_F^{(\ell)} \xrightarrow{1\times \mathrm{Id}} {}^L\mathbf{G}_\phi \xrightarrow{\xi_\varphi} {}^L\mathbf{G}$ in which $\mathbf{G}_\phi$ is a group of GL-type that splits over a tamely ramified ℓ-extension, that only depends on ϕ, and such that $\hat{\mathbf{G}}_\phi = C_{\hat{\mathbf{G}}}(\phi)$. In particular the assumption of the last conjecture is satisfied and thus the following conjecture is a consequence of the latter :

Conjecture. — There is an equivalence of categories $\mathrm{Rep}_{1,\overline{\mathbb{Z}}_\ell}(G_\phi) \xrightarrow{\sim} \mathrm{Rep}_{\phi,\overline{\mathbb{Z}}_\ell}(G)$ *that extends the transfer of irreducible $\overline{\mathbb{Q}}_\ell$-representations associated to the above $\xi_\varphi : {}^L\mathbf{G}_\phi \hookrightarrow {}^L\mathbf{G}$.*

As in the case of $\overline{\mathbb{Q}}_\ell$ coefficients, Lemma 2.4.8 tells us that conjectures 1.2.3 and 1.2.2 are actually equivalent.

1.2.4 *Tame parameters and level* 0 *blocks.* — An L-homomorphism $\xi : {}^L\mathbf{G}' \longrightarrow {}^L\mathbf{G}$ is called *tame* if its restriction $\xi_{|P_F}$ to the wild inertia subgroup P_F of W_F is trivial (which means it is $\hat{\mathbf{G}}$-conjugate to the map $p \mapsto (1,p)$). This definition also applies to L-parameters, for which $\mathbf{G}'$ is the trivial group, as well as to inertial and ℓ-inertial parameters. Note that neither $\mathbf{G}'$ nor $\mathbf{G}$ are required to be tamely ramified.

If $\phi \in \Phi_{\ell'-\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$ is a tame ℓ-inertial parameter of $\mathbf{G}$, the corresponding block $\mathrm{Rep}_\phi(G)$ in $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$ has *level* (or *depth*) 0, and conversely any level 0 block of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$ corresponds to a tame ℓ-inertial parameter. The following result is our best evidence in support of the above conjectures.

Theorem. — Let ξ be as in Conjecture 1.2.2, and suppose ξ is tame. Then there is an equivalence of categories $\mathrm{Rep}_{\phi',\overline{\mathbb{Z}}_\ell}(G') \xrightarrow{\sim} \mathrm{Rep}_{\phi,\overline{\mathbb{Z}}_\ell}(G)$*. Equivalently, let ϕ be as in Conjecture 1.2.3 and suppose ϕ is tame. Then there is an equivalence of categories* $\mathrm{Rep}_{1,\overline{\mathbb{Z}}_\ell}(G_\phi) \xrightarrow{\sim} \mathrm{Rep}_{\phi,\overline{\mathbb{Z}}_\ell}(G)$*.*

Beyond the restriction to tame parameters, what this theorem is missing at the moment is the compatibility with the transfer of $\overline{\mathbb{Q}}_\ell$-irreducible representations. This theorem is not proved in this paper. We will only show in 2.4.10 how it follows from the results in [5] where we construct equivalences in the specific cases where ξ is either an unramified automorphic induction, or a totally ramified base change. Let us also note that these cases are not obtained via Hecke algebra techniques, but by importing results from Deligne-Lusztig theory via coefficient systems on buildings.

Remark : a less precise version of the second half of the theorem is that any level 0 $\overline{\mathbb{Z}}_\ell$-block of GL_n is equivalent to the principal $\overline{\mathbb{Z}}_\ell$-block of an unramified group of type GL.

1.2.5 *A possible reduction to tame parameters.* — Here we reinterpret current work in progress by G. Chinello in our setting and show how it will imply (if successful) that Theorem 1.2.4 remains true without the word "tame". For this, we push our formalism so as to reduce the general case to the tame case in the following way. Instead of considering parameters with source I_F or $I_F^{(\ell)}$, consider the set

$$\Phi_{\mathrm{wild}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell) := \left\{ \psi : P_F \longrightarrow {}^L\mathbf{G}(\overline{\mathbb{Q}}_\ell), \exists \varphi \in \Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell), \varphi_{|P_F} = \psi \right\}.$$

To any ψ as above is attached a direct factor (no longer a block) $\mathrm{Rep}_\psi(G) := \bigoplus_{\phi_{|P_F}=\psi} \mathrm{Rep}_\phi(G)$ of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$. When ψ is trivial, $\mathrm{Rep}_\psi(G)$ is nothing but the level 0 subcategory of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$.

The same procedure as in 1.1.4 provides us with a factorization $\psi : P_F \xrightarrow{1} {}^L\mathbf{G}_\psi \xrightarrow{\xi} {}^L\mathbf{G}$ where $\mathbf{G}_\psi$ is a tamely ramified group of GL-type over F such that $\hat{\mathbf{G}}_\psi = C_{\hat{\mathbf{G}}}(\psi)$. In this setting, the map $\phi' \mapsto \xi \circ \phi'$ is a bijection

$$\{\phi' \in \Phi_{\ell'-\mathrm{inert}}(\mathbf{G}_\psi, \overline{\mathbb{Q}}_\ell) \text{ tame}\} \xrightarrow{\sim} \{\phi \in \Phi_{\ell'-\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell), \phi_{|P_F} = \psi\}$$

and moreover ξ induces an isomorphism $C_{\hat{\mathbf{G}}_\psi}(\phi') \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\xi \circ \phi')$. Therefore, Conjecture 1.2.3 implies the following one :

Conjecture. — There is an equivalence of categories $\mathrm{Rep}_1(G_\psi) \xrightarrow{\sim} \mathrm{Rep}_\psi(G)$ *that extends the transfer associated to the embedding* $\xi : {}^L\mathbf{G}_\psi \hookrightarrow {}^L\mathbf{G}$.

Conversely, if the equivalence predicted in the last conjecture exists, it has to restrict to an equivalence $\mathrm{Rep}_{\phi'}(G_\psi) \xrightarrow{\sim} \mathrm{Rep}_{\xi\circ\phi'}(G)$ for all $\phi' \in \Phi_{\ell'-\mathrm{inert}}(\mathbf{G}_\psi, \overline{\mathbb{Q}}_\ell)$. Therefore the latter conjecture, together with Conjecture 1.2.3 restricted to tame parameters, implies the full Conjecture 1.2.3. The same is true if we weaken all these statements by removing the compatibility with transfer of irreducible $\overline{\mathbb{Q}}_\ell$-representations.

Now, the point is that the weakened form of the last conjecture can be attacked by Hecke algebra techniques. Namely, the core of the problem is to exhibit an isomorphism between the Hecke algebra of a simple character (or rather, of its β-extension) of $\mathrm{GL}_n(F)$ and that of the first principal congruence subgroup of an appropriate $\mathrm{GL}_{n'}(F')$. This is exactly what Chinello is currently doing.

1.3 More general groups

Because its formulation fits well with Langlands' functoriality principle, a suitable version of Theorem 1.1.2 should hold for all L-homomorphisms. In this subsection we speculate on how it should work in an "ideal world", in which Langland's parametrization is known and satisfies some natural properties. In a forthcoming work, we will treat groups of *classical type,* meaning groups which are products of restriction of scalars of quasi-split classical groups, where all we need is available, and the desired equivalence of categories will be extracted from the work of Heiermann [9].

1.3.1 *An ideal world.* — Suppose we knew the existence of a coarse Langlands' parametrization map $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G) \longrightarrow \Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, $\pi \mapsto \varphi_\pi$, for any reductive group $\mathbf{G}$ over any p-adic field F, and suppose further that these parametrizations were compatible with parabolic induction as in [7, Conj. 5.2.2]. This means that if π is an irreducible subquotient of some parabolically induced representation $i_{M,P}^G(\sigma)$, then $\varphi_{\pi|W_F} \sim {}^L\iota \circ \varphi_{\sigma|W_F}$, where ${}^L\iota : {}^L\mathbf{M} \hookrightarrow {}^L\mathbf{G}$ is any embedding dual to $\mathbf{M} \hookrightarrow \mathbf{G}$. Then, as in the case of groups of GL-type, Bernstein's decomposition implies a decomposition

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G) = \prod_{\phi\in\Phi_{\mathrm{inert}}(\mathbf{G},\overline{\mathbb{Q}}_\ell)} \mathrm{Rep}_\phi(G)$$

where $\Phi_{\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell) \subset H^1(I_F, \hat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell))^{W_F}$ is the set of $\hat{\mathbf{G}}$-conjugacy classes of continuous sections $I_F \longrightarrow {}^L\mathbf{G}(\overline{\mathbb{Q}}_\ell)$ that admit an extension to an L-parameter in $\Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, and the direct factor category $\mathrm{Rep}_\phi(G)$ is characterized by its simple objects, which are the irreducible representations π such $\varphi_{\pi|I_F} \sim \phi$.

We note that these desiderata are now settled for quasi-split classical groups. Namely, the existence of Langlands' parametrization was obtained by Arthur for symplectic and orthogonal groups and by Mok for unitary groups, while the compatibility with parabolic induction follows from work of Moeglin for all these groups.

The main difference with the case of groups of GL-type is that $\mathrm{Rep}_\phi(G)$ is not necessarily a single Bernstein block. For example $\mathrm{Rep}_1(\mathrm{Sp}_4(F))$ contains the principal series block and a supercuspidal unipotent representation. Equivalently, the corresponding idempotent e_ϕ of $\mathfrak{Z}_{\overline{\mathbb{Q}}_\ell}(G)$ need not be primitive. Note that e_ϕ actually lies in the "stable" center $\mathfrak{Z}^{\mathrm{st}}_{\overline{\mathbb{Q}}_\ell}(G)$ defined in [7, 5.5.2], since an L-packet is either contained in $\mathrm{Rep}_\phi(G)$ or disjoint from it. However, the following example shows that e_ϕ needs not even be primitive in this stable center.

Example. Suppose $\mathbf{G} = \mathrm{SL}_2$ with p odd, and let ϕ be given by $i \in I_F \mapsto \mathrm{diag}(\varepsilon(i), 1) \in \mathrm{PGL}_2$ with ε the unique non-trivial quadratic character of I_F. Then an extension φ of ϕ to W_F has two possible shapes : either it is valued in the maximal torus of PGL_2 or it takes any Frobenius substitution to an order 2 element that normalizes non trivially this torus. In the language of [7, Def. 5.3.3], these extensions (called infinitesimal characters in *loc. cit.*) fall in two distinct inertial classes $[\varphi_{ps}] \sqcup [\varphi_0]$. Accordingly, we have a decomposition $\mathrm{Rep}_\phi(G) = \mathrm{Rep}_{[\varphi_{ps}]}(G) \times \mathrm{Rep}_{[\varphi_0]}(G)$, where $\mathrm{Rep}_{[\varphi_{ps}]}(G)$ is the block formed by (ramified) principal series associated to the character $\varepsilon \circ \mathrm{Art}_F^{-1}$ of the maximal compact subgroup of the diagonal torus of $\mathrm{SL}_2(F)$, while $\mathrm{Rep}_{[\varphi_0]}(G)$ is the category generated by cuspidal representations in the L-packet associated to φ_0. The cardinality of this L-packet is that of the centralizer of φ_0, i.e. 4, so that $\mathrm{Rep}_{[\varphi_0]}(G) = \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\{1\})^{\times 4}$. Accordingly, the idempotent $e_\phi \in \mathfrak{Z}_{\overline{\mathbb{Q}}_\ell}(G)$ decomposes as $e_\phi = e_{[\varphi_{ps}]} + e_{[\varphi_0]}$ in $\mathfrak{Z}_{\overline{\mathbb{Q}}_\ell}(G)$, with both $e_{[\varphi_{ps}]}$, $e_{[\varphi_0]}$ belonging to the "stable Bernstein center" (as in [7, 5.5.2]), showing that e_ϕ is not primitive, even in the "stable sense".

Remark.– The decomposition of $\mathrm{Rep}_\phi(G)$ in the above example can be generalized whenever the centralizer $C_{\hat{\mathbf{G}}}(\phi)$ is *not connected*. To see how, let us choose an extension φ of ϕ to W_F, and let us take up the procedure of 1.1.4. So, conjugacy under $\varphi(w)$ still endows $C_{\hat{\mathbf{G}}}(\phi)$, and also $C_{\hat{\mathbf{G}}}(\phi)^\circ$, with an action of W_F/I_F. But while the outer action $W_F/I_F \longrightarrow \mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi))$ is still independent of φ, the outer action $W_F/I_F \longrightarrow \mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi)^\circ)$ actually depends on φ. More precisely, if η_φ denotes the image of Frob by this outer action, then the set A_ϕ of all possible η_φ is a single $\pi_0(C_{\hat{\mathbf{G}}}(\phi))$-orbit inside $\mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi)^\circ)$. Now, observe that if φ, φ' are inertially equivalent in the sense of [7, Def. 5.3.3], then $\eta_\varphi = \eta_{\varphi'}$. This is because $\varphi'(\mathrm{Frob}) = z\varphi(\mathrm{Frob})$ for some z that belongs to some torus contained in $C_{\hat{\mathbf{G}}}(\phi)$. Therefore we get a decomposition

$$\mathrm{Rep}_\phi(G) = \prod_{\eta \in A_\phi} \mathrm{Rep}_{\phi,\eta}(G)$$

where $\mathrm{Rep}_{\phi,\eta}(G)$ is the "stable" Bernstein summand of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ whose irreducible objects π satisfy $\varphi_{\pi|I_F} \sim \phi$ and $\eta_\varphi = \eta$. It is plausible that the corresponding idempotents are primitive in the stable Bernstein center.

1.3.2 *The transfer problem.* — Suppose given an L-morphism $\xi : {}^L\mathbf{G}' \longrightarrow {}^L\mathbf{G}$ and an inertial parameter $\phi' \in \Phi_{\mathrm{inert}}(\mathbf{G}', \overline{\mathbb{Q}}_\ell)$ such that ξ induces an isomorphism $C_{\hat{\mathbf{G}}'}(\phi') \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$. In this generality, new issues arise on the path to a possible generalization of Theorem 1.1.2.

The first one is related to the internal structure of L-packets. Suppose temporarily that $\mathbf{G}$ and $\mathbf{G}'$ are *quasi-split*. It is then expected that the L-packet of $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G')$ associated to an extension φ' of ϕ is parametrized by irreducible representations of the component group $\pi_0(C_{\hat{\mathbf{G}}'}(\varphi')/Z(\hat{\mathbf{G}}')^{W_F})$. In our setting, ξ has to induce an isomorphism $C_{\hat{\mathbf{G}}'}(\varphi') \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\xi\circ\varphi')$.

Lemma. — If $\ker(\xi)$ *is commutative*[4], ξ *also induces an isomorphism* $Z(\hat{\mathbf{G}}')^{W_F} \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\xi)$.

Proof. Here, as usual, $C_{\hat{\mathbf{G}}}(\xi)$ is the centralizer of the image of ξ. So we clearly have $\xi(Z(\hat{\mathbf{G}}')^{W_F})\subset C_{\hat{\mathbf{G}}}(\xi)$. Moreover, since $Z(\hat{\mathbf{G}}')^{W_F}\subset C_{\hat{\mathbf{G}}'}(\phi')$, our running assumptions imply that $\xi_{|Z(\hat{\mathbf{G}}')^{W_F}}$ is injective. It remains to prove surjectivity. Again we have $C_{\hat{\mathbf{G}}}(\xi)\subset C_{\hat{\mathbf{G}}}(\phi)$, so any element of $C_{\hat{\mathbf{G}}}(\xi)$ has the form $\xi(x)$ for a unique $x\in C_{\hat{\mathbf{G}}'}(\phi')$, and we need to prove that $x\in Z(\hat{\mathbf{G}}')^{W_F}$.

Pick an extention φ' of ϕ'. Since $\mathrm{im}(\varphi')$ normalizes $\mathrm{im}(\phi')$, it also normalizes $C_{\hat{\mathbf{G}}'}(\phi')$, so that $[x,\mathrm{im}(\varphi')]\subset C_{\hat{\mathbf{G}}'}(\phi')\cap\ker(\xi)=\{1\}$. On the other hand, $[x,\hat{\mathbf{G}}']=[x,\hat{\mathbf{G}}'_{\mathrm{der}}]\subset\hat{\mathbf{G}}'_{\mathrm{der}}\cap\ker(\xi)$ which is finite. Since $\hat{\mathbf{G}}'$ is connected, it follows that $[x,\hat{\mathbf{G}}']=\{1\}$. Finally, since ${}^L\mathbf{G}'=\mathrm{im}(\varphi')\hat{\mathbf{G}}'$, we get $[x,{}^L\mathbf{G}']=\{1\}$, *i.e.* $x\in Z(\hat{\mathbf{G}}')^{W_F}$. ☐

Assume from now on that $\ker(\xi)$ is commutative. Since $Z(\hat{\mathbf{G}})^{W_F}\subset C_{\hat{\mathbf{G}}}(\xi)$, we get an embedding of $Z(\hat{\mathbf{G}})^{W_F}$ in $Z(\hat{\mathbf{G}}')^{W_F}$, whence a map

$$h_\xi:\ H^1(F,\mathbf{G}')\longrightarrow H^1(F,\mathbf{G}),$$

through Kottwitz's isomorphism [13, (6.4.1)]. Recall now that to any $\alpha\in H^1(F,\mathbf{G})$ is associated a "pure" inner form $\mathbf{G}_\alpha$ of $\mathbf{G}'$.

Expectation. — With the foregoing assumptions, for any $\alpha\in H^1(F,\mathbf{G})$ *there should exist an equivalence of categories* $\xi_*:\prod_{\beta\in h_\xi^{-1}(\alpha)}\mathrm{Rep}_{\phi'}(G'_\beta)\xrightarrow{\sim}\mathrm{Rep}_\phi(G_\alpha)$ *such that, for any irreducible* $\pi'\in\mathrm{Rep}_{\phi'}(G'_\beta)$ *we have* $\varphi_{\xi_*(\pi')}=\xi\circ\varphi_{\pi'}$

Example. Suppose $\mathbf{G}=\mathrm{SL}_2$, $\mathbf{G}'=\mathbf{U}(1)$ (norm 1 elements in a quadratic unramified extension) and $\xi:\mathbb{G}_m\rtimes W_F\longrightarrow \mathrm{PGL}_2\times W_F$ is the automorphic induction homomorphism. Then start with $\phi'=\theta: I_F\longrightarrow\mathbb{G}_m$ a character such that $\theta^{\sigma^2}=\theta$ and $\theta^\sigma\theta^{-1}$ has order >2 (with σ a Frobenius element). Then $\mathrm{Rep}_\phi(G)$ is generated by 2 distinct irreducible cuspidal representations of G. The centralizer $C_{\hat{\mathbf{G}}'}(\phi)=\hat{\mathbf{G}}'$ is connected, we have $H^1(F,\mathbf{G})=\{1\}$ while $H^1(F,\mathbf{G}')$ has 2 elements. Both pure inner forms of $\mathbf{G}'$ are isomorphic to $\mathbf{G}'$ and $\mathrm{Rep}_{\phi'}(G')$ is generated by a single irreducible representation (a character). This picture generalizes to supercuspidal L-packets constructed in [6].

Example. Suppose that p is odd, $\mathbf{G}=\mathrm{SO}_5$, $\mathbf{G}'=\mathrm{SO}_3\times\mathrm{SO}_3$ and that $\xi:\mathrm{SL}_2\times\mathrm{SL}_2\hookrightarrow\mathrm{Sp}_4$ is an isomorphism onto the centralizer of the element $\xi(1,-1)$ in Sp_4. Take $\phi':=(1,\varepsilon)$ with ε the unique quadratic non trivial character of I_F. In particular, $C_{\hat{\mathbf{G}}'}(\phi')=\mathrm{SL}_2\times\mathrm{SL}_2\xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$ is connected. We have $H^1(F,\mathbf{G})=\{\pm1\}$, $H^1(F,\mathbf{G}')=\{\pm1\}\times\{\pm1\}$ and h_ξ is the multiplication map. The category $\mathrm{Rep}_{\phi'}(G')$ is a Bernstein block coming from the maximal torus. On the other hand, $G'_{(-1,-1)}=(D^\times/F^\times)\times(D^\times/F^\times)$, with D the quaternion algebra, so that $\mathrm{Rep}_{\phi'}(G'_{(-1,-1)})$ decomposes into 4 blocks, each one generated by a quadratic character of the form $\psi\cdot\chi$ with ψ a quadratic unramified character of $D^\times$ and χ a quadratic ramified

[4]With a bit more work, we can weaken the hypothesis to : $Z((\ker(\xi)^\circ)_{\mathrm{der}})$ has order prime to p. The lemma may be true with no hypothesis at all

character of $D^\times$. Accordingly, $\mathrm{Rep}_\phi(G)$ is the sum of a Bernstein block coming from the torus and 4 supercuspidal blocks, associated to the four Langlands parameters $\varphi = \xi \circ \varphi'$ with $\varphi' = (\psi, \chi) \otimes \Delta : W_F \times \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2$ and where now ψ is a quadratic unramified character of W_F and χ a quadratic ramified character of W_F.

A second issue towards a generalization of Theorem 1.1.2 arises when we try to go to more general non quasi-split groups. A similar "expectation", involving Kottwitz' $B(G)_{\mathrm{bas}}$ instead of $H^1(F, G)$, might apply to "extended pure forms" of quasi-split groups, enabling one to reach inner forms of groups with connected center.

Another possibility is to add a suitable relevance condition. For example, consider condition

(R) An extension $\varphi' \in H^1(W_F, \hat{\mathbf{G}})$ of ϕ' is relevant for $\mathbf{G}'$ if and only if $\xi \circ \varphi'$ is relevant for $\mathbf{G}$.

The following statement seems to pass the crash-test of inner forms of linear groups :

Let ξ be as in the begining of this paragraph, and suppose further that condition (R) is satisfied. Then there is an equivalence of categories $\xi_* : \prod_{\beta \in \ker(h_\xi^{-1})} \mathrm{Rep}_{\phi'}(G'_\beta) \xrightarrow{\sim} \mathrm{Rep}_\phi(G)$ such that, for any irreducible $\pi' \in \mathrm{Rep}_{\phi'}(G'_\beta)$ we have $\varphi_{\xi_*(\pi')} = \xi \circ \varphi_{\pi'}$.

1.3.3 *The "reduction to unipotent" problem.* — Here the obvious new difficulty is that the centralizer $C_{\hat{\mathbf{G}}}(\phi)$ may not be connected. When it is connected, the same procedure as for groups of GL-type allows us to associate to ϕ an unramified group $\mathbf{G}_\phi$ with $\hat{\mathbf{G}}_\phi = C_{\hat{\mathbf{G}}}(\phi)$, together with factorization(s) of ϕ as $I_F \xrightarrow{1} {}^L\mathbf{G}_\phi \xrightarrow{\xi} {}^L\mathbf{G}$, see 2.1.2. Therefore, the natural expectation is :

Expectation. — Assume $\mathbf{G}$ quasi-split, $C_{\hat{\mathbf{G}}}(\phi)$ connected, and let ξ be as above. Then for any $\alpha \in H^1(F, \mathbf{G})$ there should exist an equivalence of categories $\xi_* : \prod_{\beta \in h_\xi^{-1}(\alpha)} \mathrm{Rep}_1(G_{\phi,\beta}) \xrightarrow{\sim} \mathrm{Rep}_\phi(G_\alpha)$ such that, for any irreducible $\pi' \in \mathrm{Rep}_{\phi'}(G'_\beta)$ we have $\varphi_{\xi_*(\pi')} = \xi \circ \varphi_{\pi'}$.

Examples. In the previous example with $\mathbf{G} = \mathrm{SL}_2$, we have $\mathbf{G}_\phi = \mathbf{U}(1)$ and the expectation is therefore satisfied. More generally, for ϕ the restriction of a tame parameter corresponding to a supercuspidal L-packet as considered in [6], the expectation holds (note that in this case, $\mathbf{G}_\phi$ is an anisotropic unramified torus). Also the last example above gives us an instance of this expectation in which $\mathbf{G} = \mathrm{SO}_5$ and $\mathbf{G}_\phi = \mathrm{SO}_3 \times \mathrm{SO}_3$.

More mysterious is the case when $C_{\hat{\mathbf{G}}}(\phi)$ is not connected. In 2.1.4 we define several non-connected reductive groups $\mathbf{G}_\phi^\tau$ over F, where τ belongs to a set $\overline{\Sigma}(\phi)$ equipped with a map to $H^1(F, \mathbf{G})$. We think that a similar statement to that above is plausible, with this collection of groups replacing the $\mathbf{G}_{\phi,\beta}$, at least when $C_{\hat{\mathbf{G}}}(\phi)$ is "quasi-split" in the sense that it is isomorphic to $C_{\hat{\mathbf{G}}}(\phi)^\circ \rtimes \pi_0$ with π_0 acting by some épinglage-preserving automorphisms.

More precisely, for these non-connected groups $\mathbf{G}_\phi^\tau$, there is a natural notion of "unipotent factor" $\mathrm{Rep}_1(G_\phi^\tau)$ of the category $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G_\phi^\tau)$. Namely a representation of G_ϕ^τ is unipotent if the restriction to $\mathbf{G}_\phi^{\tau,\circ}(F)$ is unipotent. Now, when $C_{\hat{\mathbf{G}}}(\phi)$ is "quasi-split", we expect that $\mathrm{Rep}_\phi(G)$ will be equivalent to the product of all $\mathrm{Rep}_1(G_\phi^\tau)$ with τ mapping to $1 \in H^1(F, \mathbf{G})$.

Example. Take up the example of paragraph 1.3.1 for $\mathbf{G} = \mathrm{SL}_2$. For the ϕ considered there, $C_{\hat{\mathbf{G}}}(\phi)$ is the normalizer of the diagonal torus. Our construction, explicitly detailed in 2.1.6, provides 3 groups : $\mathbf{G}_\phi^{ps} = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ and $\mathbf{G}_\phi^1 = \mathbf{G}_\phi^2 = \mathbf{U}(\mathbf{1}) \rtimes \mathbb{Z}/2\mathbb{Z}$, with the generator of $\mathbb{Z}/2\mathbb{Z}$

acting by the inverse map in each case. For $i = 1, 2$, the unipotent factor of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G^i_\phi)$ consists of representations that are trivial on $U(1)$, therefore $\mathrm{Rep}_1(G^i_\phi) = \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\{1\})^2$ and these two copies account for the supercuspidal factor $\mathrm{Rep}_{[\varphi_0]}(G) = \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\{1\})^4$. On the other hand, $\mathrm{Rep}_1(G^{ps}_\phi) = \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z})$ which is indeed equivalent to $\mathrm{Rep}_{[\varphi_{ps}]}(G)$.

We hope that the recent results of Heiermann [9] will enable us to confirm the above "expectations" for groups of classical type. In this case, the disconnected centralizers are explained by even orthogonal factors (in the last example, the groups $\mathbf{G}_\phi$ are "pure" inner forms of $\mathbf{O}_2$).

2 Details and proofs

Notation. Unless stated otherwise, $\hat{\mathbf{G}}$ and ${}^L\mathbf{G}$ will stand respectively for $\hat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$ and ${}^L\mathbf{G}(\overline{\mathbb{Q}}_\ell)$.

Given an exact sequence $H \hookrightarrow \widetilde{H} \twoheadrightarrow W$ of topological groups, we denote by $\Sigma(W, \widetilde{H})$ the set of continuous group sections $W \longrightarrow \widetilde{H}$ that split the sequence and by $\overline{\Sigma}(W, \widetilde{H})$ the set of H-conjugacy classes in $\Sigma(W, \widetilde{H})$. If we fix $\sigma \in \Sigma(W, \widetilde{H})$, conjugation by $\sigma(w)$ induces an action α_σ of W on H and a set-theoretic continuous projection $\pi_\sigma : \widetilde{H} \longrightarrow H$. Then the map $\sigma' \mapsto \pi_\sigma \circ \sigma'$ is a bijection $\Sigma(W, \widetilde{H}) \xrightarrow{\sim} Z^1_{\alpha_\sigma}(W, H)$ that descends to a bijection $\overline{\Sigma}(W, \widetilde{H}) \xrightarrow{\sim} H^1_{\alpha_\sigma}(W, H)$.

Suppose that $H = \mathbf{H}(\overline{\mathbb{Q}}_\ell)$ for some algebraic group $\mathbf{H}$. We will say that the extension $\widetilde{H}$ of W by H is *almost algebraic* if it is the pullback of an extension of some *finite* quotient of W by H. Equivalently, some finite index subgroup of W lifts to a normal subgroup $W' \subset \widetilde{H}$ that commutes with H. In this case, a section $\sigma \in \Sigma(W, \widetilde{H})$ is called *admissible* if for any such W' (equivalently, some W') the elements $\sigma(w)$, $w \in W$ are *semi-simple* in the quotient $\widetilde{H}/W'$ (which is the group of $\overline{\mathbb{Q}}_\ell$-points of an algebraic group).

For example, ${}^L\mathbf{G}$ is an almost algebraic extension of W_F by $\hat{\mathbf{G}}$, and the set $\Phi_{\mathrm{Weil}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$ of *admissible Weil parameters* (not Weil-Deligne) is the set of admissible elements in $\Sigma(W_F, {}^L\mathbf{G})$.

2.1 The centralizer and its dual F-groups

We start with a general reductive group $\mathbf{G}$ over F. We will denote by K_F a closed normal subgroup of W_F contained in I_F, and we fix $\phi : K_F \longrightarrow {}^L\mathbf{G}$ an *admissible* K_F-parameter, *i.e.* the restriction to K_F of an admissible Langlands parameter. As previously introduced, we denote by $C_{\hat{\mathbf{G}}}(\phi)$ the centralizer of $\phi(K_F)$ in $\hat{\mathbf{G}}$. By [13, Lemma 10.1.1] this is a reductive, possibly non-connected, subgroup of $\hat{\mathbf{G}}$.

2.1.1 *Extensions of ϕ to W_F.* — By hypothesis, ϕ can be extended to an admissible Weil parameter $\varphi : W_F \longrightarrow {}^L\mathbf{G}$. Because $\varphi(w)$ normalizes $\phi(K_F)$, it also normalizes $C_{\hat{\mathbf{G}}}(\phi)$ so that, letting w act by conjugation under $\varphi(w)$, we get an action

$$\alpha_\varphi : W_F/K_F \longrightarrow \mathrm{Aut}(C_{\hat{\mathbf{G}}}(\phi)).$$

Note that the restriction of this action to a finite index subgroup of W_F is by inner automorphisms of $C_{\hat{\mathbf{G}}}(\phi)$. Indeed, if F' splits $\mathbf{G}$, the action of $W_{F'}$ is through conjugation inside the normalizer $\mathcal{N} = \mathcal{N}_{\hat{\mathbf{G}}}(C_{\hat{\mathbf{G}}}(\phi))$, but $\mathcal{N}^\circ$ has finite index in $\mathcal{N}$ and acts by inner automorphisms since $\mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi))$ is discrete.

Now, if we pick another extension φ' and write $\varphi'(w) = \eta(w)\varphi(w)$ with $\eta(w) \in \hat{\mathbf{G}}$, then we compute that $\eta \in Z^1_{\alpha_\varphi}(W_F, C_{\hat{\mathbf{G}}}(\phi))$ (a 1-cocycle for the action α_φ). So we see in particular that

i) the outer action $W_F \longrightarrow \mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi))$ is independent of φ and factors over a finite quotient.

ii) the subgroup $\widetilde{C}_{\hat{\mathbf{G}}}(\phi) := C_{\hat{\mathbf{G}}}(\phi).\varphi(W_F)$ of ${}^L\mathbf{G}$ is independent of φ.

iii) the action of W_F on the center $Z(C_{\hat{\mathbf{G}}}(\phi))$ via α_φ is independent of φ and factors over a finite quotient.

Let us denote by $Z^1(W_F, \hat{\mathbf{G}})_\phi$ the set of all cocycles that extend ϕ and by $H^1(W_F, \hat{\mathbf{G}})_{[\phi]}$ the fiber of the equivalence class of ϕ in $H^1(K_F, \hat{\mathbf{G}})$. Then the map $\eta \mapsto \eta\varphi$ clearly is a bijection

$$Z^1_{\alpha_\varphi}(W_F/K_F, C_{\hat{\mathbf{G}}}(\phi)) \xrightarrow{\sim} Z^1(W_F, \hat{\mathbf{G}})_\phi$$

and it is easily checked that it induces a bijection

$$H^1_{\alpha_\varphi}(W_F/K_F, C_{\hat{\mathbf{G}}}(\phi)) \xrightarrow{\sim} H^1(W_F, \hat{\mathbf{G}})_{[\phi]}.$$

In order to see how admissibility is carried through this bijection, let us recast it in terms of sections. Consider the extension

$$C_{\hat{\mathbf{G}}}(\phi) \hookrightarrow \widetilde{C}_{\hat{\mathbf{G}}}(\phi) \twoheadrightarrow W_F$$

where the middle group is that of point ii) above and the map to W_F is induced by the projection ${}^L\mathbf{G} \longrightarrow W_F$. Quotienting by $\phi(K_F)$ we get an extension

$$C_{\hat{\mathbf{G}}}(\phi) \hookrightarrow \widetilde{C}_{\hat{\mathbf{G}}}(\phi)/\phi(K_F) \twoheadrightarrow W_F/K_F.$$

Then we have

$$\Sigma(W_F, {}^L\mathbf{G})_\phi = \Sigma(W_F, \widetilde{C}_{\hat{\mathbf{G}}}(\phi))_\phi \xrightarrow{\sim} \Sigma(W_F/K_F, \widetilde{C}_{\hat{\mathbf{G}}}(\phi))$$

where the index ϕ means "extends ϕ" and the last map takes a section φ to (φ mod $\phi(K_F)$). The next lemma shows that $\widetilde{C}_{\hat{\mathbf{G}}}(\phi)$, and hence $\widetilde{C}_{\hat{\mathbf{G}}}(\phi)/\phi(K_F)$, is "almost algebraic" in the sense of the begining of this section, and also that admissibility is preserved through the last bijection, giving a bijection

$$\Phi_{\mathrm{Weil}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)_{[\phi]} \xrightarrow{\sim} \overline{\Sigma}(W_F/K_F, \widetilde{C}_{\hat{\mathbf{G}}}(\phi)/\phi(K_F))_{\mathrm{adm}}.$$

Lemma. — There exist a finite extension F' of F and an extension $\varphi' : W_F \longrightarrow {}^L\mathbf{G}$ of ϕ such that $\varphi'(w') = (1, w')$ for all $w' \in W_{F'}$.

Proof. Start with any extension φ as above. Let F_0 be a finite Galois extension of F that splits $\mathbf{G}$ and denote by $\bar{\varphi}$ the composition of φ with the projection ${}^L\mathbf{G} \longrightarrow \hat{\mathbf{G}} \rtimes \Gamma_{F_0/F}$. Then $\bar{\varphi}(W_F)$ is an extension of a cyclic (possibly infinite) group by a finite group and there is some finite extension F' of F such that $\bar{\varphi}(W_{F'})$ is central in $\bar{\varphi}(W_F)$ and $\bar{\varphi}(I_{F'})$ is trivial. We may also assume that F' contains F_0, so that $\bar{\varphi}(W_{F'}) \subset \hat{\mathbf{G}}$ and actually $\bar{\varphi}(W_{F'}) \subset C_{\hat{\mathbf{G}}}(\bar{\varphi}) = C_{\hat{\mathbf{G}}}(\varphi)$. Enlarging F' further, we may assume that $\bar{\varphi}(W_{F'}) \subset C_{\hat{\mathbf{G}}}(\varphi)^\circ$. Now, since W_F/I_F is cyclic and contains $W_{F'}/I_{F'}$ with finite index, we can find a homomorphism $\chi : W_F/I_F \longrightarrow C_{\hat{\mathbf{G}}}(\varphi)^\circ$ such that $\chi_{|W_{F'}} = \bar{\varphi}_{|W_{F'}}$. Then $\varphi' := \chi^{-1}\varphi$ has the desired property. $\square$

2.1.2 *The group $\mathbf{G}_\phi$ in the connected case.* — Let us assume in this paragraph that $C_{\hat{\mathbf{G}}}(\phi)$ is *connected*, and let $\mathbf{G}_\phi^{\mathrm{split}}$ denote a dual group for $C_{\hat{\mathbf{G}}}(\phi)$ defined over F. By item i) of 2.1.1, we have a canonical outer action $W_F/K_F \longrightarrow \mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi)) = \mathrm{Out}(\mathbf{G}_\phi^{\mathrm{split}})$ that factors over a finite quotient. Choosing a section $\mathrm{Out}(\mathbf{G}_\phi^{\mathrm{split}}) \longrightarrow \mathrm{Aut}(\mathbf{G}_\phi^{\mathrm{split}})$ (i.e. choosing an épinglage of $\mathbf{G}_\phi^{\mathrm{split}}$), we get an action of the Galois group Γ_F on $\mathbf{G}_\phi^{\mathrm{split}}$. Associated to this action is an F-form, $\mathbf{G}_\phi$ of $\mathbf{G}_\phi^{\mathrm{split}}$, which by construction is a quasi-split connected reductive group over F, uniquely defined up to F-isomorphism, and which splits over an extension F' such that $K_{F'} = K_F$.

For any continous cocycle $\tau : \Gamma_F \longrightarrow \mathbf{G}_\phi(\overline{F})$, we have an inner form $\mathbf{G}_\phi^\tau$ of $\mathbf{G}_\phi$ over F. If τ' is cohomologous to τ, there is an F-isomorphism $\mathbf{G}_\phi^{\tau'} \xrightarrow{\sim} \mathbf{G}_\phi^\tau$ well-defined up to inner automorphism. We will therefore identify $\mathbf{G}_\phi^\tau$ and $\mathbf{G}_\phi^{\tau'}$, and we now have a collection $(\mathbf{G}_\phi^\tau)_{\tau\in H^1(F,\mathbf{G}_\phi)}$ of F-groups associated to ϕ.

In order to shrink this collection, we now define a map $H^1(F,\mathbf{G}_\phi) \xrightarrow{h_\phi} H^1(F,\mathbf{G})$ by using the Kottwitz isomorphism. Recall the latter is an isomorphism $H^1(F,\mathbf{G}) \xrightarrow{\sim} \pi_0(Z(\hat{\mathbf{G}})^{W_F})^*$ with $*$ denoting a Pontrjagin dual. In the case of $\mathbf{G}_\phi$, this reads $H^1(F,\mathbf{G}_\phi) \xrightarrow{\sim} \pi_0(Z(C_{\hat{\mathbf{G}}}(\phi))^{W_F})^*$, with W_F acting through the canonical action of point iii) in 2.1.1. Now the desired map is induced by the inclusion $Z(\hat{\mathbf{G}})^{W_F} \subset Z(C_{\hat{\mathbf{G}}}(\phi))^{W_F}$.

The role of this map is the following. If the group $\mathbf{G}$ is quasi-split, the factor category $\mathrm{Rep}_\phi(G)$ is expected to be equivalent to the direct product of the unipotent factors of each $\mathbf{G}_\phi^\tau$ for $\tau \in \ker(h_\phi)$. More generally, for a pure inner form $\mathbf{G}^\alpha$ of a quasi-split $\mathbf{G}$ associated to some $\alpha \in H^1(F,\mathbf{G})$, the factor category $\mathrm{Rep}_\phi(G^\alpha)$ is expected to be equivalent to the direct product of the unipotent factors of each $\mathbf{G}_\phi^\tau$ for $h_\phi(\tau) = \alpha$.

All these connected reductive F-groups $\mathbf{G}_\phi^\tau$ share the same L-group, which we denote by ${}^L\mathbf{G}_\phi$. As usual, it is defined "up to inner automorphism". To fix ideas, let us choose an épinglage ε of $C_{\hat{\mathbf{G}}}(\phi)$. Then, as a model for the L-group we can take

$$ {}^L\mathbf{G}_\phi = C_{\hat{\mathbf{G}}}(\phi) \rtimes_{\alpha_\phi^\varepsilon} W_F \tag{2.1.3}$$

where the action α_ϕ^ε is obtained from the canonical outer action via the splitting $\mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi)) \hookrightarrow \mathrm{Aut}(C_{\hat{\mathbf{G}}}(\phi))$ associated to ε. The L-group ${}^L\mathbf{G}_\phi$ is an extension of W_F by $C_{\hat{\mathbf{G}}}(\phi)$ but it is not a priori clear whether it is isomorphic to the extension $\widetilde{C}_{\hat{\mathbf{G}}}(\phi)$. More precisely, let $\widetilde{\mathcal{N}}(\varepsilon)$ be the stabilizer of ε in $\widetilde{C}_{\hat{\mathbf{G}}}(\phi)$ acting by conjugacy on $C_{\hat{\mathbf{G}}}(\phi)$. Then $\widetilde{\mathcal{N}}(\varepsilon)$ is an extension of W_F by the center $Z(C_{\hat{\mathbf{G}}}(\phi))$, and we see that

Lemma. — *The extensions $\widetilde{C}_{\hat{\mathbf{G}}}(\phi)$ and ${}^L\mathbf{G}_\phi$ of W_F by $C_{\hat{\mathbf{G}}}(\phi)$ are isomorphic if and only if the class $[\widetilde{\mathcal{N}}(\varepsilon)]$ in $H^2(W_F/K_F, Z(C_{\hat{\mathbf{G}}}(\phi)))$ vanishes.*

Note that the class always vanishes if $K_F = I_F$ since $W_F/I_F = \mathbb{Z}$. We give more details in section 2.2.

2.1.4 *More F-groups in the general case.* — We now propose a construction without assuming that $C_{\hat{\mathbf{G}}}(\phi)$ is connected. Let us take up the split exact sequence

$$C_{\hat{\mathbf{G}}}(\phi) \hookrightarrow \widetilde{C}_{\hat{\mathbf{G}}}(\phi)/\phi(K_F) \twoheadrightarrow W_F/K_F$$

of paragraph 2.1.1. Put $\pi_0(\phi) := \pi_0(C_{\hat{\mathbf{G}}}(\phi))$ and $\widetilde{\pi}_0(\phi) := \widetilde{C}_{\hat{\mathbf{G}}}(\phi)/C_{\hat{\mathbf{G}}}(\phi)^\circ\phi(K_F)$. Hence we have a split exact sequence

$$\pi_0(\phi) \hookrightarrow \widetilde{\pi}_0(\phi) \twoheadrightarrow W_F/K_F$$

and a possibly non split exact sequence

$$C_{\hat{\mathbf{G}}}(\phi)^\circ \hookrightarrow \widetilde{C}_{\hat{\mathbf{G}}}(\phi)/\phi(K_F) \twoheadrightarrow \widetilde{\pi}_0(\phi).$$

Conjugation by any set-theoretic section of the above sequence gives a well-defined "outer action"

$$\widetilde{\pi}_0(\phi) \longrightarrow \mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi)^\circ)$$

that factors over a finite quotient of $\widetilde{\pi}_0(\phi)$. Let $\mathbf{G}_\phi^{\mathrm{split},\circ}$ denote a split group over F which is dual to $C_{\hat{\mathbf{G}}}(\phi)^\circ$, and fix a section

$$\mathrm{Out}(C_{\hat{\mathbf{G}}}(\phi)^\circ) = \mathrm{Out}(\mathbf{G}_\phi^{\mathrm{split},\circ}) \longrightarrow \mathrm{Aut}(\mathbf{G}_\phi^{\mathrm{split},\circ})$$

(i.e. fix an épinglage of $\mathbf{G}_\phi^{\mathrm{split},\circ}$). Then we can form the non-connected reductive F-group

$$\mathbf{G}_\phi^{\mathrm{split}} := \mathbf{G}_\phi^{\mathrm{split},\circ} \rtimes \pi_0(\phi)$$

which has an action by algebraic F-group automorphisms

$$\widetilde{\pi}_0(\phi) \xrightarrow{\theta} \mathrm{Aut}(\mathbf{G}_\phi^{\mathrm{split}})$$

(here $\widetilde{\pi}_0(\phi)$ acts on $\pi_0(\phi)$ by conjugation). Then, any continuous section

$$\sigma : W_F/K_F \longrightarrow \widetilde{\pi}_0(\phi)$$

(for the topology induced from Γ_F on W_F and the discrete topology on $\pi_0(\phi)$) provides an F-form $\mathbf{G}_\phi^\sigma$ of $\mathbf{G}_\phi^{\mathrm{split}}$ such that the action of W_F on $\mathbf{G}_\phi^\sigma(\overline{F}) = \mathbf{G}_\phi^{\mathrm{split}}(\overline{F})$ is the natural action twisted by $\theta \circ \sigma$. The unit component $\mathbf{G}_\phi^{\sigma,\circ}$ is quasi-split over F and we have

$$\mathbf{G}_\phi^\sigma(F) = \mathbf{G}_\phi^{\sigma,\circ}(F) \rtimes \pi_0(\phi)^{\sigma(W_F)}.$$

Moreover, if $c \in \pi_0(\phi)$, conjugation by $(1, c)$ in $\mathbf{G}_\phi^{\mathrm{split}}(\overline{F})$ induces an F-isomorphism $\mathbf{G}_\phi^\sigma \xrightarrow{\sim} \mathbf{G}_\phi^{\sigma^c}$, so that the isomorphism class of $\mathbf{G}_\phi^\sigma$ over F only depends on the image of σ in $\overline{\Sigma}(W_F/K_F, \widetilde{\pi}_0(\phi))$.

More generally, for any continuous section

$$\tau : W_F/K_F \longrightarrow \mathbf{G}_\phi^{\mathrm{split},\circ}(\overline{F}^{K_F}) \rtimes \widetilde{\pi}_0(\phi)$$

(again, here we use the topology of W_F induced from that of Γ_F) we get an F-form $\mathbf{G}_\phi^\tau$ of $\mathbf{G}_\phi^{\mathrm{split}}$, which depends only on the image of τ in $\overline{\Sigma}(W_F/K_F, \mathbf{G}_\phi^{\circ,\mathrm{split}}(\overline{F}^{K_F}) \rtimes \widetilde{\pi}_0(\phi))$. In order to better organize this collection of F-groups, we use the projection $\tau \mapsto \sigma$:

$$\overline{\Sigma}\left(W_F/K_F, \mathbf{G}_\phi^{\mathrm{split},\circ}(\overline{F}^{K_F}) \rtimes \widetilde{\pi}_0(\phi)\right) \longrightarrow \overline{\Sigma}\left(W_F/K_F, \widetilde{\pi}_0(\phi)\right)$$

to get a partition

$$\overline{\Sigma}(\phi) := \overline{\Sigma}\left(W_F/K_F, \mathbf{G}_\phi^{\mathrm{split},\circ}(\overline{F}^{K_F}) \rtimes \widetilde{\pi}_0(\phi)\right) = \bigsqcup_{\sigma\in\overline{\Sigma}(W_F,\widetilde{\pi}_0(\phi))} H^1\left(W_F/K_F, \mathbf{G}_\phi^{\sigma,\circ}(\overline{F}^{K_F})\right)$$
$$= \bigsqcup_{\sigma\in\overline{\Sigma}(W_F,\widetilde{\pi}_0(\phi))} H^1\left(F, \mathbf{G}_\phi^{\sigma,\circ}\right)$$

For the second line, we use that $H^1\left(W_F/K_F, \mathbf{G}_\phi^{\sigma,\circ}(\overline{F}^{K_F})\right) = H^1\left(W_F, \mathbf{G}_\phi^{\sigma,\circ}(\overline{F})\right)$ due to the fact that $H^1(K_F, \mathbf{G}^{\sigma,\circ}(\overline{F})) = \{1\}$, and we use that $H^1\left(W_F, \mathbf{G}_\phi^{\sigma,\circ}(\overline{F})\right) = H^1\left(\Gamma_F, \mathbf{G}_\phi^{\sigma,\circ}(\overline{F})\right)$ due to our non-standard choice of topology on W_F.

Hence, if τ is mapped to σ, the group $\mathbf{G}_\phi^\tau$ is an inner form of $\mathbf{G}_\phi^\sigma$ "coming from the unit component". However, the collection of all $\mathbf{G}_\phi^\tau$'s should be viewed as a single "pure inner class" of (possibly non-connected) reductive groups. As in the case of connected centralizers, we need a map

$$\overline{\Sigma}(\phi) \longrightarrow H^1(F, \mathbf{G})$$

in order to shrink this collection of groups. We define it as the coproduct of the maps $H^1(F, \mathbf{G}_\phi^{\sigma,\circ}) \longrightarrow H^1(F, \mathbf{G})$ which are dually induced by inclusions

$$Z(\hat{\mathbf{G}})^{W_F} \subset Z(C_{\hat{\mathbf{G}}}(\phi)^\circ)^{\widetilde{\pi}_0(\phi)} \subset Z(C_{\hat{\mathbf{G}}}(\phi)^\circ)^{\sigma(W_F)} = Z(\widehat{\mathbf{G}_\phi^{\sigma,\circ}})^{W_F}.$$

Finally we define a common L-group for all these non-connected groups. Namely we put

$$(2.1.5) \qquad {}^L\mathbf{G}_\phi := (C_{\hat{\mathbf{G}}}(\phi)^\circ \rtimes_{\alpha_\phi^\varepsilon} \widetilde{\pi}_0(\phi)) \times_{W_F/K_F} W_F$$

where ε is an épinglage, and α_ϕ^ε is the action obtained from the canonical outer action thanks to this épinglage. The main inconvenience of this L-group is that it might not be always isomorphic to the extension $\widetilde{C}_{\hat{\mathbf{G}}}(\phi)$. More precisely, let $\widetilde{\mathcal{N}}(\varepsilon)$ be the stabilizer of ε in $\widetilde{C}_{\hat{\mathbf{G}}}(\phi)$ acting by conjugacy on $C_{\hat{\mathbf{G}}}(\phi)$. Then $\widetilde{\mathcal{N}}(\varepsilon)$ is an extension of $\widetilde{\pi}_0 \times_{W_F/K_F} W_F$ by the center $Z(C_{\hat{\mathbf{G}}}(\phi)^\circ)$, and we see that

Lemma. — The extensions $\widetilde{C}_{\hat{\mathbf{G}}}(\phi)$ and ${}^L\mathbf{G}_\phi$ of $\widetilde{\pi}_0 \times_{W_F/K_F} W_F$ by $C_{\hat{\mathbf{G}}}(\phi)^\circ$ are isomorphic if and only if the class $[\widetilde{\mathcal{N}}(\varepsilon)]$ in $H^2(\widetilde{\pi}_0(\phi), Z(C_{\hat{\mathbf{G}}}(\phi))^\circ)$ vanishes.

Therefore, a necessary condition for ${}^L\mathbf{G}_\phi \simeq \widetilde{C}_{\hat{\mathbf{G}}}(\phi)$ is that $\pi_0(\phi)$ has a lifting in $C_{\hat{\mathbf{G}}}(\phi)$ which fixes ε. When $K_F = I_F$, this condition is sufficient since $W_F/I_F = \mathbb{Z}$.

2.1.6 *An example.* — Assume p is odd, $K_F = I_F$ and let ε denote the unique non-trivial quadratic continuous character of I_F. Then consider the group $\mathbf{G} = \mathrm{SL}_2$ and the parameter $\phi : I_F \longrightarrow \mathrm{PGL}_2(\overline{\mathbb{Q}}_\ell) \times W_F$ that takes i to $\left(\begin{pmatrix} \varepsilon(i) & 0 \\ 0 & 1 \end{pmatrix}, i\right)$. Then $C_{\hat{\mathbf{G}}}(\phi)$ is the normalizer $\hat{\mathbf{N}}$ of the diagonal torus $\hat{\mathbf{T}}$ of $\hat{\mathbf{G}}$. Denote by s the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, of order 2. We thus have $\mathbf{G}_\phi^{\mathrm{split}} = \mathbb{G}_m \rtimes \{1, s\}$ with s acting by the inverse map. Moreover we have $\widetilde{\pi}_0(\phi) = \{1, s\} \times W_F/I_F$, so that $\overline{\Sigma}(W_F/K_F, \widetilde{\pi}_0(\phi))$ has 2 elements σ_0, σ_1, with σ_0 the trivial morphism. Clearly we have $\mathbf{G}_\phi^{\sigma_0} = \mathbf{G}_\phi^{\mathrm{split}} = \mathbb{G}_m \rtimes \{1, s\}$. On the other hand we have $\mathbf{G}_\phi^{\sigma_1} = \mathbf{U}(1) \rtimes \{1, s\}$ where $\mathbf{U}(1)$ is the group of norm 1 elements in the unramified quadratic extension of F and s again acts by the inverse map. Further, $\overline{\Sigma}(\phi) = H^1(F, \mathbb{G}_m) \sqcup H^1(F, \mathbf{U}(1))$ has 3 elements, $\tau_0, \tau_{1,1}$ and $\tau_{1,2}$. We compute that

$$\mathbf{G}_\phi^{\tau_0} = \mathbb{G}_m \rtimes \{1, s\}, \quad \text{and} \quad \mathbf{G}_\phi^{\tau_{1,1}} = \mathbf{G}_\phi^{\tau_{1,2}} = \mathbf{U}(1) \rtimes \{1, s\}.$$

Observe also that in this case we have ${}^L\mathbf{G}_\phi \simeq \widetilde{C}_{\hat{\mathbf{G}}}(\phi)$.

2.2 Unipotent factorizations of a K_F-parameter

We keep the general setup of the previous section, consisting of a connected reductive F-group $\mathbf{G}$ and an admissible parameter $\phi : K_F \longrightarrow {}^L\mathbf{G}$. We have associated an L-group ${}^L\mathbf{G}_\phi$ to ϕ, see (2.1.3) and (2.1.5), after fixing an épinglage ε of $C_{\hat{\mathbf{G}}}(\phi)^\circ$. We will denote by $1 : K_F \longrightarrow {}^L\mathbf{G}_\phi$ the morphism that takes $k \in K_F$ to $(1, k)$.

2.2.1 *Definition.–* A *strict* unipotent factorization of ϕ is a morphism of L-groups $\xi : {}^L\mathbf{G}_\phi \longrightarrow {}^L\mathbf{G}$ that extends the inclusion $C_{\hat{\mathbf{G}}}(\phi)^\circ \hookrightarrow \hat{\mathbf{G}}$ and satisfies $\xi \circ 1 = \phi$. Two such factorizations are called equivalent if they are conjugate by some element $\hat{g} \in \hat{\mathbf{G}}$.

2.2.2 *Proposition. — Suppose that $C_{\hat{\mathbf{G}}}(\phi)$ is* connected. *In the following statements, we use the canonical action of W_F/K_F on $Z(C_{\hat{\mathbf{G}}}(\phi))$.*

i) The map $\xi \mapsto \varphi := \xi_{|W}$ sets up a bijection between

$$\{\textit{strict unipotent factorizations of } \phi\} \textit{ and}$$

$$\{\textit{parameters } \varphi : W_F \longrightarrow {}^L\mathbf{G} \textit{ that extend } \phi \textit{ and such that } \alpha_\varphi \textit{ preserves } \varepsilon.\}$$

ii) Multiplication of cocycles turns the second set of point i) into a torsor over $Z^1(W_F/K_F, Z(\hat{\mathbf{G}}_\phi))$.

iii) The set of equivalence classes of strict unipotent factorizations of ϕ is a torsor over $H^1(W_F/K_F, Z(C_{\hat{\mathbf{G}}}(\phi)))$.

iv) There is an obstruction element $\beta_\phi \in H^2(W_F/K_F, Z(C_{\hat{\mathbf{G}}}(\phi)))$ which vanishes if and only if ϕ admits a strict unipotent factorization.

Proof. i) From the equality $\xi(\hat{c}, w) = \hat{c}.\varphi(w)$, we see that the map is well-defined. To prove it is a bijection it suffices to check that the inverse map $\varphi \mapsto \xi : (\hat{c}, w) \mapsto \hat{c}.\varphi(w)$ is well-defined too. But any extension φ of ϕ to W_F leads to a factorization

$$\phi : \; K_F \xrightarrow{i \mapsto (1,i)} C_{\hat{\mathbf{G}}}(\phi) \rtimes_{\alpha_\varphi} W_F \xrightarrow{(\hat{c},w) \mapsto \hat{c}.\varphi(w)} {}^L\mathbf{G}\,. \tag{2.2.3}$$

If φ preserves the épinglage ε, then the semi-direct product in the middle is ${}^L\mathbf{G}_\phi$ and the map on the right hand side is therefore a strict unipotent factorization.

ii) Suppose the second set of point i) is not empty, and let φ be an element in this set. Then any other φ' in this set has the form $\varphi'(w) = \eta(w)\varphi(w)$ for some $\eta \in Z^1_{\alpha_\varphi}(W_F, C_{\hat{\mathbf{G}}}(\phi))$. Since $\varphi_{|K_F} = \varphi'_{|K_F}$ we have in fact $\eta \in Z^1_{\alpha_\varphi}(W_F/K_F, C_{\hat{\mathbf{G}}}(\phi))$. Moreover, since both α_φ and

$\alpha_{\varphi'}$ preserve ϵ and induce the same outer automorphism, $\mathrm{Int}_{\eta(w)}$ has to be trivial, that is, $\eta \in Z^1_{\alpha_\varphi}(W_F/K_F, Z(C_{\hat{\mathbf{G}}}(\phi))) = Z^1(W_F/K_F, Z(C_{\hat{\mathbf{G}}}(\phi)))$ (the action on $Z(C_{\hat{\mathbf{G}}}(\phi))$ is canonical). This shows that, if non empty, the second set in point i) is a (obviously principal) homogeneous set over $Z^1(W_F/K_F, Z(C_{\hat{\mathbf{G}}}(\phi)))$.

iii) Suppose both ξ and $\xi' = \mathrm{Int}_{\hat{g}} \circ \xi$, with $\hat{g} \in \hat{\mathbf{G}}$, are strict unipotent factorizations of ϕ. Since $\xi_{|K_F} = \xi'_{|K_F} = \phi$, we must have $\hat{g} \in C_{\hat{\mathbf{G}}}(\phi)$. But since also $\xi_{|C_{\hat{\mathbf{G}}}(\phi)} = \xi'_{|C_{\hat{\mathbf{G}}}(\phi)} = \phi$, we have $\hat{g} \in Z(C_{\hat{\mathbf{G}}}(\phi))$. Then we see that $\xi'_{|W} = \eta.\xi_{|W}$, where η is the boundary cocycle $w \mapsto (\hat{g}.{}^w(\hat{g})^{-1})$. Hence point iii) follows from ii) and i).

iv) This is Lemma 2.1.2. Here is a more detailed argument. Start with an arbitrary extension φ of ϕ to W_F. We need to investigate the existence of a cocycle $\eta \in Z^1_{\alpha_\varphi}(W_F/K_F, C_{\hat{\mathbf{G}}}(\phi))$ such that $\eta\varphi$ fixes the épinglage ε. This is equivalent to asking that $\alpha^\varepsilon_\phi(w) = \mathrm{Int}_{\eta(w)} \circ \alpha_\varphi(w)$ for all $w \in W_F$. But for $w \in W_F/K_F$, there is a unique element $\beta^\varepsilon_\varphi(w) \in C_{\hat{\mathbf{G}}}(\phi)_{\mathrm{ad}}$ such that $\alpha^\varepsilon_\phi(w) = \mathrm{Ad}_{\beta^\varepsilon_\varphi(w)} \circ \alpha_\varphi(w)$. Unicity insures that the map $w \mapsto \beta^\varepsilon_\varphi(w)$ lies in $Z^1_{\alpha_\varphi}(W_F/K_F, C_{\hat{\mathbf{G}}}(\phi)_{\mathrm{ad}})$, and the existence of η as above is equivalent to the vanishing of the image β_ϕ of $\beta^\varepsilon_\varphi$ by the boundary map

$$H^1_{\alpha_\varphi}(W_F/K_F, C_{\hat{\mathbf{G}}}(\phi)_{\mathrm{ad}}) \longrightarrow H^2(W_F/K_F, Z(C_{\hat{\mathbf{G}}}(\phi))).$$

$\square$

2.2.4 *Remark.* — Here is the significance of point iii) in terms of transfer of representations. Assume that a strict unipotent factorization ξ of ϕ exists. Then the transfer map dual to ξ, from the set of L-packets of G_ϕ to that of G, only depends on the $Z(\hat{\mathbf{G}}_\phi)$-conjugacy class of ξ. If we change ξ to $\eta.\xi$ for $\eta \in H^1(W_F/K_F, Z(\hat{\mathbf{G}}_\phi))$ then the transfer map is twisted by the character of G_ϕ associated to η in [2, 10.2] (this character is unramified if $K_F = I_F$ or has level 0 for $K_F = P_F$).

2.2.5 *Corollary.* — *When $K_F = I_F$, any parameter ϕ admits a strict unipotent factorization (provided $C_{\hat{\mathbf{G}}}(\phi)$ is connected).*

Proof. In this case $W_F/K_F = \mathbb{Z}$, so $H^2(W_F/K_F, A) = 0$ for any $\mathbb{Z}[W_F/K_F]$-module A. $\square$

2.2.6 *Remark.* — In the non-connected case, points iii) and iv) remain true with the pair $(W_F/K_F, Z(C_{\hat{\mathbf{G}}}(\phi)))$ replaced by $(\widetilde{\pi}_0(\phi), Z(C_{\hat{\mathbf{G}}}(\phi)^\circ))$. However, we will not use it in this paper.

In the sequel, we will also encounter "non-strict" unipotent factorizations.

2.2.7 *Definition.–* A *unipotent factorization* is a pair $(\mathbf{H}, \xi)$ consisting of a *connected* reductive F-group and a morphism of L-groups ${}^L\mathbf{H} \xrightarrow{\xi} {}^L\mathbf{G}$ such that

- ϕ is $\hat{\mathbf{G}}$-conjugate to $\xi \circ 1$ with 1 the trivial parameter $k \in K_F \mapsto (1, k) \in {}^L\mathbf{H}$.
- ξ induces an isomorphism $\hat{\mathbf{H}} \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$.

Let us make explicit the relation between unipotent factorizations as above and strict ones. If ι is an épinglage-preserving W_F-equivariant isomorphism $\hat{\mathbf{H}} \xrightarrow{\sim} \hat{\mathbf{G}}_\phi$, we denote by ${}^L\iota := \iota \times \mathrm{Id}_{W_F} : {}^L\mathbf{H} \xrightarrow{\sim} {}^L\mathbf{G}_\phi$ the associated isomorphism of L-groups.

2.2.8 *Proposition.* — *If $(\mathbf{H},\xi)$ is a unipotent factorization of ϕ, there are an épinglage-preserving W_F-equivariant isomorphism $\iota : \hat{\mathbf{H}} \xrightarrow{\sim} \hat{\mathbf{G}}_\phi$ and a strict unipotent factorization ξ' such that ξ is $\hat{\mathbf{G}}$-conjugate to $\xi' \circ {}^L\iota$. Moreover ι is unique and ξ' is unique up to equivalence.*

Proof. Conjugating ξ under $\hat{\mathbf{G}}$ we may assume that $\xi \circ 1 = \phi$. Conjugating further ξ under $C_{\hat{\mathbf{G}}}(\phi)$, we may assume also that ξ takes the given épinglage on $\hat{\mathbf{H}}$ to ε. Now let φ' be the trivial parameter $W_F \longrightarrow {}^L\mathbf{H}$ and put $\varphi := \xi \circ \varphi'$. Then $\xi_{|\hat{\mathbf{H}}}$ is W_F-equivariant for the actions $\alpha_{\varphi'}$ on $\hat{\mathbf{H}}$ and α_φ on $C_{\hat{\mathbf{G}}}(\phi)$. But $\alpha_{\varphi'}$ is the natural action on $\hat{\mathbf{H}}$ and preserves the given épinglage, hence α_φ preserves ε. Therefore $\iota := \xi_{|\hat{\mathbf{H}}}$ is an épinglage-preserving W_F-equivariant isomorphism $\hat{\mathbf{H}} \xrightarrow{\sim} \hat{\mathbf{G}}_\phi$ and $\xi' := \xi \circ {}^L\iota^{-1}$ is a strict unipotent factorization, whence the existence statement. Unicity of ι is clear, and any other ξ'' has to be both $\hat{\mathbf{G}}_\phi$-conjugate to ξ' and strict, hence is $Z(\hat{\mathbf{G}}_\phi)$-conjugate to ξ'. $\square$

Remark.– We see in particular that a general unipotent factorization $(\mathbf{G}_\phi, \xi')$ is equivalent to the composition of a strict one with an "outer" W_F-equivariant automorphism of $\hat{\mathbf{G}}_\phi$.

2.3 Restriction of scalars

We consider here a reductive group $\mathbf{G}$ over F of the form $\mathbf{G} = \mathrm{Res}_{F'|F}\mathbf{G}'$ for some reductive group $\mathbf{G}'$ over some extension field F' of F. We then have the following relationship between their dual groups equipped with Weil group actions :

$$\hat{\mathbf{G}} = \mathrm{Ind}_{W_{F'}}^{W_F} \hat{\mathbf{G}}' = \left\{ \hat{g} : W_F \longrightarrow \hat{\mathbf{G}}', \forall (w', w) \in W_{F'} \times W_F, \hat{g}(w'w) = {}^{w'}(\hat{g}(w)) \right\}$$

where we have denoted with an exponent ${}^{w'}$ the action of $W_{F'}$ on $\hat{\mathbf{G}}'$ and we let $v \in W_F$ act on $\mathbf{G}$ by $(v\hat{g})(w) := \hat{g}(wv)$.

2.3.1 We still denote by K_F a closed normal subgroup of W_F, and we put $K_{F'} := W_{F'} \cap K_F$. Note that if K_F is one of the groups I_F, $I_F^{(\ell)}$ or P_F, we have respectively $K_{F'} = I_{F'}$, $I_{F'}^{(\ell)}$ or $P_{F'}$. There is a natural map on continuous cocycles

$$Z^1(K_F, \hat{\mathbf{G}}) \longrightarrow Z^1(K_{F'}, \hat{\mathbf{G}}')$$

that takes the cocycle $(\hat{g}_\gamma)_{\gamma \in K_F}$ to the cocycle $(\hat{g}_{\gamma'}(1))_{\gamma' \in K_{F'}}$. We will call it the "Shapiro map". It is compatible with coboundary relation thus induces a map

$$H^1(K_F, \hat{\mathbf{G}}) \longrightarrow H^1(K_{F'}, \hat{\mathbf{G}}').$$

Shapiro's lemma asserts that when $K_F = W_F$ the map on Z^1 is onto while the map on H^1 is a bijection. From the definition of Shapiro's map we have two commutative diagrams

$$\begin{array}{ccc} Z^1(W_F, \hat{\mathbf{G}}) & \twoheadrightarrow & Z^1(W_{F'}, \hat{\mathbf{G}}') \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ Z^1(K_F, \hat{\mathbf{G}}) & \longrightarrow & Z^1(K_{F'}, \hat{\mathbf{G}}') \end{array} \quad \text{and} \quad \begin{array}{ccc} H^1(W_F, \hat{\mathbf{G}}) & \xrightarrow{\sim} & H^1(W_{F'}, \hat{\mathbf{G}}') \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ H^1(K_F, \hat{\mathbf{G}}) & \longrightarrow & H^1(K_{F'}, \hat{\mathbf{G}}') \end{array}$$

2.3.2 *Lemma.* — *The right hand square is cartesian.*

Proof. The two diagrams are transitive with respect to intermediate field extensions. Applying this to the extension F'' defined by $W_{F''} = W_{F'}K_F$, we see it is enough to prove the claim in the following 2 cases : a) $K_F = K_{F'}$ or b) $W_F = K_F W_{F'}$.

In case b), we have $\hat{\mathbf{G}} = \mathrm{Ind}_{K_{F'}}^{K_F} \hat{\mathbf{G}}'$ as groups with K_F-action, so that the bottom map of our diagram is also an isomorphism and the claim is clear.

In case a) we have to prove that for any two cocycles $(\hat{g}_w)_{w\in W_F}$, and $(\hat{h}_w)_{w\in W_F}$ in $Z^1(W_F, \hat{\mathbf{G}})$, we have

$$\left(\forall\gamma \in K_F = K_{F'},\, \hat{g}_\gamma(1) = \hat{h}_\gamma(1)\right) \Rightarrow \left((\hat{g}_\gamma)_{\gamma\in K_F} = (\hat{h}_\gamma)_{\gamma\in K_F} \text{ in } H^1(K_F, \hat{\mathbf{G}})\right).$$

Let us fix a set $\{v_1 = 1, \dots, v_r\}$ of representatives of left $W_{F'}$-cosets in W_F and let us denote by $\hat{k}$ the unique element of $\hat{\mathbf{G}}$ such that $\hat{k}(v_i) = \hat{h}_{v_i}(1)^{-1}\hat{g}_{v_i}(1)$ for all i.

The cocycle property tells us that $\hat{g}_\gamma(v) = \hat{g}_v(1)^{-1}\hat{g}_{v\gamma}(1)$ for all $v \in W_F$, hence also $\hat{g}_\gamma(v) = \hat{g}_v(1)^{-1} \cdot \hat{g}_{v\gamma v^{-1}}(1) \cdot {}^{v\gamma v^{-1}}(\hat{g}_v(1))$ and the same for $\hat{h}$. Since by hypothesis we have $\hat{g}_{v\gamma v^{-1}}(1) = \hat{h}_{v\gamma v^{-1}}(1)$, this implies that for each i we have

$$\hat{h}_\gamma(v_i) = \hat{k}(v_i) \cdot \hat{g}_\gamma(v_i) \cdot {}^{v_i\gamma v_i^{-1}}\hat{k}(v_i)^{-1} = \hat{k}(v_i) \cdot \hat{g}_\gamma(v_i) \cdot \hat{k}(v_i\gamma)^{-1}$$

Then, for any $v \in W_F$, writing uniquely $v = v'v_i$ with $v' \in W_{F'}$ we get

$$\hat{h}_\gamma(v) = {}^{v'}\hat{h}_\gamma(v_i) = {}^{v'}\hat{k}(v_i) \cdot {}^{v'}\hat{g}_\gamma(v_i) \cdot {}^{v'}\hat{k}(v_i\gamma)^{-1} = \hat{k}(v) \cdot \hat{g}_\gamma(v) \cdot \hat{k}(v\gamma)^{-1}.$$

Since $\hat{k}(v\gamma) = (\gamma\hat{k})(v)$, this shows that $(\hat{h}_\gamma)_{\gamma\in K_F}$ is cohomologous to $(\hat{g}_\gamma)_{\gamma\in K_F}$. □

Recall that, by definition, the set of admissible K_F-parameters for $\mathbf{G}$ is the set of continous sections $\phi : K_F \longrightarrow {}^L\mathbf{G}$ such that, writing $\phi(\gamma) = (\hat{\phi}_\gamma, \gamma)$, we have

$$(\hat{\phi}_\gamma)_{\gamma\in K_F} \in \mathrm{Image}\left(\Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\text{res}} H^1(W_F, \hat{\mathbf{G}}) \xrightarrow{\text{res}} H^1(K_F, \hat{\mathbf{G}})\right).$$

Recall also that we have denoted this set by $\Phi_{\text{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, $\Phi_{\ell'-\text{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$ and $\Phi_{\text{wild}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$ according to K_F being I_F, $I_F^{(\ell)}$ and P_F.

2.3.3 *Corollary.* — *The Shapiro map induces bijections* $\Psi(\mathbf{G}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \Psi(\mathbf{G}', \overline{\mathbb{Q}}_\ell)$ *where* Ψ *denotes either* Φ_{inert} *or* $\Phi_{\ell'-\text{inert}}$ *or* Φ_{wild}.

Proof. This is because the Shapiro bijection $H^1(W_F, \hat{\mathbf{G}}) \longrightarrow H^1(W_{F'}, \hat{\mathbf{G}}')$ preserves the admissibility conditions on both sides [2, 8.4]. □

Let $\phi : K_F \longrightarrow {}^L\mathbf{G}$ be an admissible K_F-parameter and let ϕ' be its Shapiro mate. Pick a parameter $\varphi : W_F \longrightarrow {}^L\mathbf{G}$ that extends ϕ and let φ' be its Shapiro mate. We get an action $\mathrm{Int}_\varphi : w \mapsto \mathrm{Int}_{\varphi(w)}$ of W_F on $\hat{\mathbf{G}}$, where $\mathrm{Int}_{\varphi(w)}$ means conjugation by $\varphi(w)$ inside ${}^L\mathbf{G}$. Similarly, we have an action $\mathrm{Int}_{\varphi'}$ of $W_{F'}$ on $\hat{\mathbf{G}}'$. Now consider the map

$$\begin{array}{rcl} (\hat{\mathbf{G}}, \mathrm{Int}_\varphi) & \to & \mathrm{Ind}_{W_{F'}}^{W_F}(\hat{\mathbf{G}}', \mathrm{Int}_{\varphi'}) \\ \hat{g} & \mapsto & \tilde{g} : w \mapsto [\mathrm{Int}_{\varphi(w)}(\hat{g})](1) \end{array}$$

It is easily checked that this map is well-defined and is a W_F-equivariant isomorphism of groups. In fact, writing $\varphi(w) = (\hat{\varphi}_w, w)$, we have $\tilde{g}(w) = \hat{\varphi}_w(1)\hat{g}(w)\hat{\varphi}_w(1)^{-1}$, so that the inverse isomorphism is given by $\hat{g}(w) = \hat{\varphi}_w(1)^{-1}\tilde{g}(w)\hat{\varphi}_w(1)$.

The centralizer $C_{\hat{\mathbf{G}}}(\phi)$ is stable under the action Int_φ, and in the last section we had denoted the resulting action by α_φ. Similarly, $C_{\hat{\mathbf{G}}}(\phi')$ is stable under the action $\mathrm{Int}_{\varphi'}$.

2.3.4 *Lemma. — The above isomorphism takes $C_{\hat{\mathbf{G}}}(\phi)$ into $\mathrm{Ind}_{W_{F'}}^{W_F}(C_{\hat{\mathbf{G}}'}(\phi'))$ and induces an isomorphism*

$$(C_{\hat{\mathbf{G}}}(\phi), \alpha_\varphi) \xrightarrow{\sim} \mathrm{Ind}_{W_{F'}/K_{F'}}^{W_F/K_F}(C_{\hat{\mathbf{G}}'}(\phi'), \alpha_{\varphi'})$$

where we identify the RHS with the K_F-invariant functions in $\mathrm{Ind}_{W_{F'}}^{W_F}(C_{\hat{\mathbf{G}}'}(\phi'))$. Moreover, α_φ preserves an épinglage of $C_{\hat{\mathbf{G}}}(\phi)^\circ$ if and only if $\alpha_{\varphi'}$ preserves an épinglage of $C_{\hat{\mathbf{G}}'}(\phi')^\circ$.

Proof. By definition $C_{\hat{\mathbf{G}}}(\phi)$ is the subgroup of fixed points under K_F acting on $(\hat{\mathbf{G}}, \mathrm{Int}_\varphi)$. Hence the above isomorphism carries it to the subgroup $\mathrm{Ind}_{W_{F'}}^{W_F}(\hat{\mathbf{G}}', \varphi')^{K_F}$. However for a function $\tilde{g}$, being K_F-invariant means $\tilde{g}(w\gamma) = \tilde{g}(w)$ for all $w \in W_F$ and $\gamma \in K_F$. Since K_F is normal in W_F this is equivalent to $\tilde{g}(\gamma w) = \tilde{g}(w)$ for all w, γ. Applying this to $\gamma \in K_{F'}$ we get that $\tilde{g}(w) \in C_{\hat{\mathbf{G}}'}(\phi')$ for all w, as claimed.

Now if $\varepsilon = (B, T, \{x_\alpha\})$ is an épinglage of $C_{\hat{\mathbf{G}}}(\phi)$ fixed by α_φ, evaluation at 1 in the isomorphism of the lemma provides an épinglage of $C_{\hat{\mathbf{G}}'}(\phi')$ fixed by $\alpha_{\varphi'}$. Conversely, let $\varepsilon' = (B', T', (x')_{\alpha' \in \Delta'})$ be an épinglage of $C_{\hat{\mathbf{G}}'}(\phi')$ stable by $\alpha_{\varphi'}$. Put $B = \mathrm{Ind}_{W_{F'}/K_{F'}}^{W_F/K_F}(B')$ and $T = \mathrm{Ind}_{W_{F'}/K_{F'}}^{W_F/K_F}(T')$. This is a Borel pair in the group $\mathrm{Ind}_{W_{F'}/K_{F'}}^{W_F/K_F}(C_{\hat{\mathbf{G}}'}(\phi'))$, with set of simple roots $\Delta = \mathrm{Ind}(\Delta') = (W_{F'}/K_{F'})\backslash[(W_F/K_F) \times \Delta']$. For a simple root $\alpha = \overline{(v, \alpha')}$, let $x_\alpha : W_F/K_F \longrightarrow \mathrm{Hom}(\mathbb{G}_a, C_{\hat{\mathbf{G}}'}(\phi'))$ be the function supported on $W_{F'}v$ given by $x_\alpha(w'v) = {}^{w'}x_{\alpha'} = x_{w'\alpha'}$. The triple $(B, T, (x_\alpha)_{\alpha \in \Delta})$ is then a W_F-stable épinglage of $\mathrm{Ind}_{W_{F'}/K_{F'}}^{W_F/K_F}(C_{\hat{\mathbf{G}}'}(\phi'))$ which, through the isomorphism of the lemma, provides an épinglage ε of $C_{\hat{\mathbf{G}}}(\phi)$ fixed by α_φ. ☐

Suppose now that $C_{\hat{\mathbf{G}}'}(\phi')$ is connected, or equivalently, that $C_{\hat{\mathbf{G}}}(\phi)$ is connected, and let us fix épinglages ε and ε' to build the L-groups ${}^L\mathbf{G}_\phi$ and ${}^L\mathbf{G}'_{\phi'}$. The following is a translation of the last lemma in the language of the previous section.

2.3.5 *Corollary. — i) We have the following relation between $\mathbf{G}_\phi$ and $\mathbf{G}'_{\phi'}$. Denote by F'' the intermediate extension such that $W_{F''} = W_{F'}K_F$ and let $\phi'' : K_{F''} \longrightarrow {}^L\mathbf{G}''$ with $\mathbf{G}'' = \mathrm{Res}_{F'|F''}\mathbf{G}'$ be the Shapiro mate of ϕ'. Then we have*

$$\mathbf{G}_\phi \simeq \mathrm{Res}_{F''|F}\mathbf{G}''_{\phi''} \quad \textit{and} \quad \mathbf{G}'_{\phi'} \simeq \mathbf{G}''_{\phi''} \times_{F''} F',$$

whence in particular an L-homomorphism (unique up to conjugacy)

$${}^L\mathbf{G}_\phi \xrightarrow{\xi_u} {}^L\mathrm{Res}_{F'|F}(\mathbf{G}'_{\phi'})$$

*which is an isomorphism if $F'' = F'$ (*i.e. $K_{F'} = K_F$*), while its adjoint ${}^L(\mathbf{G}_\phi \times_F F') \longrightarrow {}^L\mathbf{G}'_{\phi'}$ is an isomorphism if $F'' = F$ (*i.e $W_F = W_{F'}K_F$*).*

ii) ϕ *admits a strict unipotent factorization if and only if* ϕ' *does. Moreover, if* $\xi : {}^L\mathbf{G}_\phi \longrightarrow {}^L\mathbf{G}$ *is a strict unipotent factorization of* ϕ*, there are a strict unipotent factorization* $\xi' : {}^L\mathbf{G}'_{\phi'} \longrightarrow {}^L\mathbf{G}'$ *of* ϕ' *and a factorization*

$$\xi : {}^L\mathbf{G}_\phi \xrightarrow{\xi_u} {}^L\mathrm{Res}_{F'|F}(\mathbf{G}'_{\phi'}) \xrightarrow{\tilde{\xi}'} {}^L\mathrm{Res}_{F'|F}(\mathbf{G}') = {}^L\mathbf{G}$$

with $\tilde{\xi}'_{|\widehat{\mathrm{Res}(\mathbf{G}'_{\phi'})}} = \mathrm{Ind}_{W_{F'}}^{W_F}(\xi'_{|\hat{\mathbf{G}}'_{\phi'}})$ *and* $\tilde{\xi}'_{|W_F}$ *a Shapiro lift of* $\xi'_{|W_{F'}}$*.*

2.3.6 *Remark.* — Because of the form taken by $\tilde{\xi}'$, the tranfer map

$$\tilde{\xi}'_* : \Phi(\mathrm{Res}_{F'|F}(\mathbf{G}'_{\phi'}), \overline{\mathbb{Q}}_\ell) \longrightarrow \Phi(\mathrm{Res}_{F'|F}(\mathbf{G}'), \overline{\mathbb{Q}}_\ell)$$

coincides with the transfer map ξ'_* through the Shapiro bijections. Moreover, since ξ_u induces an isomorphism $\hat{\mathbf{G}}_\phi^{K_F} \xrightarrow{\sim} \widehat{\mathrm{Res}_{F'|F}\mathbf{G}'_{\phi'}}^{K_F}$, the map $\tilde{\xi}'$ induces an isomorphism $C(1) = \widehat{\mathrm{Res}_{F'|F}\mathbf{G}'_{\phi'}}^{K_F} \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$.

2.4 Groups of GL-type

Recall that $\mathbf{G}$ is of GL-type if it is isomorphic to a product of groups of the form $\mathrm{Res}_{F'|F}(\mathrm{GL}_n)$. For such a group, the local Langlands correspondence for GL_n and the Shapiro lemma provide a bijection $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(F)) \xrightarrow{\sim} \Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, $\pi \mapsto \varphi_\pi$.

2.4.1 *Lemma.* — *Let* $\phi \in \Phi_{\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$*. Define* $\mathrm{Rep}_\phi(G)$ *as the smallest direct factor of* $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ *which contains all irreducible* π *such that* $\varphi_{\pi|I_F} \sim \phi$*. Then* $\mathrm{Rep}_\phi(G)$ *is a Bernstein block of* $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$*.*

Proof. We may assume that $\mathbf{G} = \mathrm{Res}_{F'|F}(\mathrm{GL}_n)$. In this case, Lemma 2.3.2 and Corollary 2.3.3 provide us with a Shapiro bijection $\Phi_{\mathrm{inert}}(\mathrm{GL}_n, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \Phi_{\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, $\phi' \mapsto \phi$, such that $\mathrm{Rep}_{\phi'}(\mathrm{GL}_n(F')) = \mathrm{Rep}_\phi(\mathbf{G}(F))$. So we are reduced to the case $\mathbf{G} = \mathrm{GL}_n$. In this case, we need to prove that the extensions φ of ϕ fall in a single "inertial class". Equivalently, writing $\varphi(w) = (\hat{\varphi}(w), w)$ with $\hat{\varphi}$ an n-dimensional representation, we see that we need to prove that if two semisimple representations $\hat{\varphi}, \hat{\varphi}'$ of W_F are isomorphic as I_F-representations, then there are decompositions $\hat{\varphi} = \hat{\varphi}_1 \oplus \cdots \oplus \hat{\varphi}_r$ and $\hat{\varphi}' = \hat{\varphi}'_1 \oplus \cdots \oplus \hat{\varphi}'_r$ and unramified characters χ_i, $i = 1, \ldots, r$, of W_F such that $\hat{\varphi}'_i = \chi_i \hat{\varphi}_i$ for all $i = 1, \cdots, r$. But Clifford theory tells us that the restriction of an irreducible $\hat{\varphi}$ to I_F has multiplicity one, and that any W_F-invariant multiplicity one semisimple representations of I_F extends to a representation of W_F which is irreducible and unique up to unramified twist. Hence any decomposition $\hat{\varphi} = \hat{\varphi}_1 \oplus \cdots \oplus \hat{\varphi}_r$ into irreducible summands has to be matched by a similar decomposition of $\hat{\varphi}'$ satisfying the desired twisting property. □

Therefore, we have a parametrization of Bernstein blocks of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(F))$ by $\Phi_{\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, which moreover is compatible with the Shapiro bijection. Let us turn to Vignéras-Helm blocks.

2.4.2 *Proposition.* — *Let* $\phi \in \Phi_{\ell'-\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$*. There is a unique direct factor subcategory* $\mathrm{Rep}_\phi(G)$ *of* $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\phi)$ *such that for any* $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$ *we have* $\pi \in \mathrm{Rep}_\phi(G)$ *if and only if* $\varphi_{\pi|I_F^{(\ell)}} \sim \phi$*. Moreover* $\mathrm{Rep}_\phi(G)$ *is a block.*

Proof. Again, we may assume that $\mathbf{G} = \mathrm{Res}_{F'|F}(\mathrm{GL}_n)$, and using Lemma 2.3.2 and Corollary 2.3.3, we are reduced to the case $\mathbf{G} = \mathrm{GL}_n$. The Vignéras blocks of $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\mathrm{GL}_n(F))$ are parametrized by inertial classes of semisimple $\overline{\mathbb{F}}_\ell$-representations of W_F, and the same proof as above shows these are in bijection with isomorphism classes of semisimple $\overline{\mathbb{F}}_\ell$-representations of I_F that extend to W_F. As explained in 1.2.1, the latter are in bijection with isomorphism classes of (semisimple) $\overline{\mathbb{Q}}_\ell$-representations of $I_F^{(\ell)}$, via reduction mod ℓ and restriction. Going through these identifications, considering the definition of the "mod ℓ inertial supercuspidal support" of a $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n(F))$ in [10, Def. 4.10] and applying Theorem 11.8 of [10], we find that $\pi, \pi' \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n(F))$ lie in the same Helm block if and only if φ_π and $\varphi_{\pi'}$ have isomorphic restrictions to $I_F^{(\ell)}$. □

We thus get a parametrization of Vignéras-Helm blocks of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(F))$ by $\Phi_{\ell'-\mathrm{inert}}(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, which is compatible with Shapiro bijections.

2.4.3 *Assumptions and convention.* — In the sequel, K_F will denote one of the subgroups I_F, $I_F^{(\ell)}$ or P_F of W_F. The notation $\mathrm{Rep}_\phi(G)$ will denote a block of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$ if $K_F = I_F^{(\ell)}$, and a block of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ if $K_F = I_F$. When $K_F = P_F$, it will denote a direct factor of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$. On the other hand, the notation $\mathrm{Irr}_\phi(G)$ will always denote the set of $\overline{\mathbb{Q}}_\ell$-irreducible representations in this block.

2.4.4 We will denote by $\mathcal{E}_F(\phi', \xi)$ the following statement, that depends on an admissible K_F-parameter $\phi' : K_F \longrightarrow {}^L\mathbf{G}'$ and an L-homomorphism $\xi : {}^L\mathbf{G}' \longrightarrow {}^L\mathbf{G}$ which induces an isomorphism $C_{\hat{\mathbf{G}}'}(\phi') \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$, where $\phi = \xi \circ \phi'$.

$$\mathcal{E}_F(\phi', \xi) : \left| \begin{array}{l} \text{there is an equivalence of categories } \mathrm{Rep}_{\phi'}(G') \xrightarrow{\sim} \mathrm{Rep}_\phi(G) \\ \text{that extends the transfer map } \xi_* : \mathrm{Irr}_{\phi'}(G') \longrightarrow \mathrm{Irr}_\phi(G). \end{array} \right.$$

We also denote by $\mathcal{E}_F(\phi', \xi)^-$ the same statement without the condition on the transfer map.

2.4.5 *Example.* — Suppose that ξ is a Levi subgroup embedding. Then we can embed $\mathbf{G}'$ as an F-rational Levi subgroup of $\mathbf{G}$ (well-defined up to conjugacy). The assumption that ξ induces an isomorphism of centralizers translates into the property that the normalizer of the inertial supercuspidal support of any $\pi \in \mathrm{Irr}_\phi(G)$ is contained (up to conjugacy) in $\mathbf{G}'$. In this context, it is known that for any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with $\mathbf{G}'$ as a Levi component, the *normalized* parabolic induction functor Ind_P^G induces an equivalence of categories $\mathrm{Rep}_\phi(G') \xrightarrow{\sim} \mathrm{Rep}_\phi(G)$ (which is independent of the choice of P up to natural transform). We refer to [10, Thm. 12.3] for $\overline{\mathbb{Z}}_\ell$ coefficients. We claim that this equivalence is compatible with the transfer ξ_*. Indeed, using the Shapiro yoga, it is enough to treat the case of $\mathbf{G} = \mathrm{GL}_n$ and $\mathbf{G}' = \prod_{i=1}^r \mathrm{GL}_{n_i}$, where ξ_* is given in terms of representations by $(\sigma_1, \cdots, \sigma_r) \mapsto \sigma_1 \oplus \cdots \oplus \sigma_r$, which is well known to correspond to normalized parabolic induction in this context. Therefore $\mathcal{E}_F(\phi', \xi)$ is satisfied in this setting.

2.4.6 *Computation of* $\mathbf{G}_\phi$ *when* $\mathbf{G} = \mathrm{GL}_n$. — In this case we may write $\phi = \hat{\phi} \times \mathrm{Id}_{W_F}$ where $\hat{\phi}$ is an n-dimensional semi-simple representation of K_F that can be extended to W_F. Our aim is to find a nice extension $\hat{\varphi}$ of $\hat{\phi}$ to W_F. There is a decomposition $\hat{\phi} = \hat{\phi}_1 \oplus \cdots \oplus \hat{\phi}_r$, uniquely determined (up to reordering) by the following properties :

i) the irreducible constituents of $\hat{\phi}_i$ form a W_F-orbit,

ii) $\mathrm{Hom}_{K_F}(\hat{\phi}_i, \hat{\phi}_j) = 0$ whenever $i \neq j$.

Since this decomposition is preserved by any extension of $\hat{\phi}$ to W_F, each $\hat{\phi}_i$ is extendable to W_F. Putting $n_i := \dim \hat{\phi}_i$, this means that ϕ factors through a Levi subgroup embedding $\iota : (\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}) \times W_F \hookrightarrow \mathrm{GL}_n \times W_F$. Moreover, by ii) this Levi subgroup contains the centralizer of ϕ (in other words ι induces an isomorphism of centralizers).

To compute the centralizer, let us write

$$\hat{\phi}_i = \overline{\mathbb{Q}}_\ell^{e_i} \otimes \left(\bigoplus_{w \in W_F/W_{\sigma_i}} {}^w\sigma_i \right)$$

where σ_i is some irreducible representation of K_F and W_{σ_i} its stabilizer in W_F. This decomposition identifies $C_{\hat{\mathbf{G}}}(\phi)$ with $\prod_{i=1}^r \mathrm{GL}_{e_i}(\overline{\mathbb{Q}}_\ell)^{[W_F:W_{\sigma_i}]}$.

Lemma. — Any irreducible representation σ of K_F can be extended to its normalizer W_σ in W_F.

Proof. The case $K_F = I_F$ is clear since $W_F/I_F \simeq \mathbb{Z}$, so we assume that $K_F \subsetneq I_F$. In this case, W_σ/K_F is the semi-direct product of a pro-cyclic group $(W_\sigma \cap I_F)/K_F$ by a copy of $\mathbb{Z}$ acting by multiplication by q^r for some $r > 0$. Hence σ can be extended to a representation $\widetilde{\sigma}_0$ of $W_\sigma \cap I_F$ such that the generator w of $\mathbb{Z}$ acts by ${}^w\widetilde{\sigma}_0 \simeq \widetilde{\sigma}_0\chi$ for some character χ of $(W_\sigma \cap I_F)/K_F$. But this character admits a $(q^r-1)^{\mathrm{th}}$-root χ_0 (since $x \mapsto x^{q^r-1}$ is surjective on μ_{ℓ^∞} and $\mu_{p'^\infty}$), so that $\widetilde{\sigma}_0\chi_0^{-1}$ is fixed by w, hence extends to a representation $\widetilde{\sigma}$ as desired. ☐

Let us apply this lemma to each σ_i and pick an extension $\tilde{\sigma}_i$ to W_{σ_i}. Then, putting

$$\hat{\varphi} := \hat{\varphi}_1 \oplus \cdots \oplus \hat{\varphi}_r, \quad \text{with } \hat{\varphi}_i = \overline{\mathbb{Q}}_\ell^e \otimes \mathrm{Ind}_{W_{\sigma_i}}^{W_F}(\tilde{\sigma}_i),$$

we get an extension φ of ϕ such that

$$(C_{\hat{\mathbf{G}}}(\phi), \alpha_\varphi) \simeq \prod_{i=1}^r \mathrm{Ind}_{W_{\sigma_i}}^{W_F}(\mathrm{GL}_{e_i})$$

where W_{σ_i} acts trivially on GL_{e_i}. In particular α_φ fixes any diagonal épinglage of $C_{\hat{\mathbf{G}}}(\phi) \simeq \prod_{i=1}^r \mathrm{GL}_{e_i}^{[W_F:W_{\sigma_i}]}$ and we may identify $C_{\hat{\mathbf{G}}}(\phi) \rtimes_{\alpha_\varphi} W_F$ with the L-group ${}^L\mathbf{G}_\phi$. Denoting by F_i the finite extension such that $W_{F_i} = W_{\sigma_i}$, we then see that $\mathbf{G}_\phi \simeq \prod_{i=1}^r \mathrm{Res}_{F_i|F}(\mathrm{GL}_{e_i})$, and that the factorization (2.2.3) is a unipotent factorization $\xi = \xi_\varphi$ of ϕ of the following form

$$\xi : {}^L\mathbf{G}_\phi = \left(\prod_{i=1}^r \mathrm{Ind}_{W_{F_i}}^{W_F} \mathrm{GL}_{e_i} \right) \rtimes W_F \longrightarrow \left(\prod_{i=1}^r \mathrm{GL}_{n_i} \right) \times W_F \hookrightarrow \mathrm{GL}_n \times W_F,$$

where $\xi_{|W_F} = \varphi$.

2.4.7 *Proposition. — Let ϕ be a K_F-parameter of a group $\mathbf{G}$ of GL-type. Then $\mathbf{G}_\phi$ is also of GL-type and ϕ admits a strict unipotent factorization $\phi : K_F \xrightarrow{1 \times \mathrm{Id}} {}^L\mathbf{G}_\phi \xrightarrow{\xi} {}^L\mathbf{G}$.*

Proof. By definition of being of GL-type, we may assume that $\mathbf{G} = \mathrm{Res}_{F'|F}\mathbf{G}'$ for $\mathbf{G}' = \mathrm{GL}_n$ and F' a finite separable extension of F. By Corollary 2.3.5, we may assume that $\mathbf{G} = \mathrm{GL}_n$, which has just been treated. $\square$

This proposition allows us to consider the following statement, that depends on an admissible parameter $\phi :\ K_F \longrightarrow {}^L\mathbf{G}$ (with $\mathbf{G}$ unspecified) :

$$\mathcal{U}_F(\phi) : \left| \begin{array}{l} \text{for any strict unipotent factorization } \xi : {}^L\mathbf{G}_\phi \longrightarrow {}^L\mathbf{G}, \\ \text{there is an equivalence of categories } \mathrm{Rep}_1(G_\phi) \xrightarrow{\sim} \mathrm{Rep}_\phi(G) \\ \text{that extends the transfer map } \xi_* : \mathrm{Irr}_1(G_\phi) \longrightarrow \mathrm{Irr}_\phi(G). \end{array} \right.$$

Note, that because of Remark 2.2.4, replacing "any" by "one" in the first line gives an equivalent statement. Again, we will denote by $\mathcal{U}_F(\phi)^-$ the same statement without the compatibility with transfer.

2.4.8 *Lemma. — The following are equivalent.*

i) Statement $\mathcal{E}_F(\phi', \xi)$ is true for all F, ϕ' and ξ satisfying the required conditions.

ii) Statement $\mathcal{U}_F(\phi)$ is true for all F and ϕ.

iii) Statement $\mathcal{U}_F(\phi)$ is true for all F and ϕ pertaining to $\mathbf{G} = \mathrm{GL}_n$, and statement $\mathcal{E}_F(1, \xi)$ is true for all base change $\xi :\ {}^L\mathrm{GL}_n \longrightarrow {}^L\mathrm{Res}_{F'|F}\mathrm{GL}_n$, with $W_{F'}K_F = W_F$ (and F allowed to vary).

Moreover the same equivalence holds for statements $\mathcal{E}_F(\phi', \xi)^-$ and $\mathcal{U}_F(\phi)^-$.

Proof. *i)* $\Rightarrow$ *iii)* is clear. To prove *ii)* $\Rightarrow$ *i)*, start with (ϕ', ξ), choose a strict unipotent factorization ξ' of ϕ' and consider the diagram

$$\phi :\ K_F \xrightarrow{1} {}^L\mathbf{G}'_{\phi'} \xrightarrow{\xi'} {}^L\mathbf{G}' \xrightarrow{\xi} {}^L\mathbf{G}.$$

Then $\xi \circ \xi'$ is a unipotent factorization of ϕ, albeit not strict a priori. By Proposition 2.2.8, it is equivalent to the composition $\xi'' \circ \alpha$ of a strict unipotent factorization and a W_F-invariant outer automorphism α of $\hat{\mathbf{G}}_\phi$. A W_F-invariant outer automorphism of $\hat{\mathbf{G}}_\phi$ induces an F-automorphism of $\mathbf{G}_\phi$, well-defined up to G_ϕ-conjugacy, hence an endo-equivalence of categories of $\mathrm{Rep}(G_\phi)$ and in particular of $\mathrm{Rep}_1(G_\phi)$ (since the trivial representation is fixed). By [7, Prop. 5.2.5], this equivalence is known to be compatible with Langlands' transfer. Therefore, using this equivalence and the ones granted by $\mathcal{U}_F(\phi')$ and $\mathcal{U}_F(\phi)$, we get $\mathcal{E}(\phi', \xi)$.

Let us prove *iii)* $\Rightarrow$ *ii)*. We want to check $\mathcal{U}_F(\phi)$ for any ϕ. It is sufficient to do so when $\mathbf{G} = \mathrm{Res}_{F'|F}(\mathrm{GL}_n)$. Let ξ be a strict unipotent factorization of ϕ. We have a factorization $\xi = \tilde{\xi}' \circ \xi_u$ as in Corollary 2.3.5 ii). By hypothesis, and thanks to Remark 2.3.6, we can find an equivalence of categories associated to $\tilde{\xi}'$, so we are left with finding one associated to ξ_u. With the notation of Corollary 2.3.5 i), we have a further factorization of ξ_u :

$$\xi_u : {}^L\mathbf{G}_\phi \xrightarrow{\sim} {}^L\mathrm{Res}_{F''|F}(\mathbf{G}''_{\phi''}) \longrightarrow {}^L\mathrm{Res}_{F'|F}(\mathbf{G}'_{\phi'})$$

which shows that it is sufficient to do it when $F'' = F$, *i.e.* $W_{F'}K_F = W_F$. In this case, ξ_u is a base change L-homomorphism $\xi_u :\ {}^L\mathbf{G}_\phi \longrightarrow {}^L\mathrm{Res}_{F'|F}(\mathbf{G}_\phi \times_F F')$. Now, $\mathbf{G}_\phi$ is of GL-type and

"K_F-unramified", in the sense that it splits over an extension F_0 of F such that $K_{F_0} = K_F$. So we need an equivalence associated to a base change homomorphism of the form

$$\xi_u : {}^L(\mathrm{Res}_{F_0|F}\mathrm{GL}_n) \longrightarrow {}^L\mathrm{Res}_{F'|F}(\mathrm{Res}_{F_0|F}\mathrm{GL}_n \times_F F')$$

where F_0 is a K_F-unramified extension of F. But then F' and F_0 are disjoint, so we have $\mathrm{Res}_{F_0|F}\mathrm{GL}_n \times_F F' = \mathrm{Res}_{F'F_0|F'}\mathrm{GL}_n$ and the above L-homomorphism takes the form

$${}^L(\mathrm{Res}_{F_0|F}\mathrm{GL}_n) \longrightarrow {}^L\mathrm{Res}_{F_0|F}(\mathrm{Res}_{F_0F'|F_0}\mathrm{GL}_n)$$

and is thus "induced" from the base change L-homomorphism over F_0

$${}^L(\mathrm{GL}_n) \longrightarrow {}^L\mathrm{Res}_{F_0F'|F_0}\mathrm{GL}_n.$$

Using Remark 2.3.6 again, it is enough to associate an equivalence to the latter L-homomorphism, but this is precisely part of the hypothesis in iii). $\square$

2.4.9 *Proof of Theorem 1.1.2.* — Here we assume that $K_F = I_F$ and we will prove that $\mathcal{E}_F(\phi', \xi)$ holds true for all ϕ' and ξ that satisfy the required conditions. It is enough to prove the statements in point iii) of the previous lemma. We will denote by $\mathcal{H}(q, n)$ the extended Iwahori-Hecke algebra of type A_{n-1} with parameter q over $\overline{\mathbb{Q}}_\ell$. We will also denote by q_F the cardinality of the residue field of F.

Totally ramified base change of the unipotent block of GL_n. We use Zelevinski's classification $m \mapsto Z(m)$ of $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_n(F))$ in terms of multisegments of unramified characters of $F^\times$. Let $F'|F$ be a totally ramified extension. The base change for GL_1 is induced by the norm map $(F')^\times \longrightarrow F^\times$. Since the latter induces an isomorphism $F'^\times/\mathcal{O}_{F'}^\times \xrightarrow{\sim} F^\times/\mathcal{O}_F^\times$, the base change is a bijection on unramified characters, hence also on multisegments $m \mapsto m'$. Since base change is compatible with parabolic induction, and thus with the Langlands quotient construction, it is also compatible with the Zelevinski construction, in the sense that the base change of $Z(m)$ has to be $Z(m')$. Now, by a theorem of Borel, there is a natural equivalence of categories between $\mathrm{Mod}(\mathcal{H}(q_F, n))$ and $\mathrm{Rep}_1(\mathrm{GL}_n(F))$ which takes Rogawski's classification of simple modules of $\mathcal{H}(q_F, n)$ in terms of multisegments of characters of $\mathbb{Z}^n$ to Zelevinski's classification, see [14]. The desired equivalence between $\mathrm{Rep}_1(\mathrm{GL}_n(F))$ and $\mathrm{Rep}_1(\mathrm{GL}_n(F'))$ hence follows from the equality $q_{F'} = q_F$.

Property $\mathcal{U}_F(\phi)$ *for* GL_n. As in paragraph 2.4.6, let us write $\phi = \hat{\phi} \times \mathrm{Id}_{I_F}$ and decompose $\hat{\phi} = \hat{\phi}_1 \oplus \cdots \oplus \hat{\phi}_r$. If $r > 1$, paragraph 2.4.6 tells us that any unipotent factorization of ϕ factors through a Levi embedding that induces an isomorphism of centralizers. Thanks to Example 2.4.5, we may thus assume $r = 1$. In this case, let σ be an irreducible constituent of $\hat{\phi}$. Its stabilizer W_σ in W_F is the Weil group W_{F_f} of "the" unramified extension of degree $f = [W_F : W_\sigma]$ and paragraph 2.4.6 tells us that $\mathbf{G}_\phi \simeq \mathrm{Res}_{F_f|F}(\mathrm{GL}_e)$ where $e = n/(f\mathrm{dim}\sigma)$. Pick an extension $\tilde{\sigma}$ of σ to W_{F_f} and put $\hat{\varphi} := \overline{\mathbb{Q}}_\ell^e \otimes \mathrm{Ind}_{W_{F_f}}^{W_F}(\tilde{\sigma})$. We get an extension φ of ϕ to W_F, whose associated strict unipotent factorization ξ_φ has the following effect on parameters. Identify $\Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, resp. $\Phi(\mathbf{G}_\phi, \overline{\mathbb{Q}}_\ell)$, with the set of (classes of) Frobenius-semisimple continuous $\overline{\mathbb{Q}}_\ell$-representations of W'_F of dimension n, resp. of W'_{F_f} of dimension e. Then the transfer map $\xi_{\varphi,*}$ is given by

$$\rho \in \Phi(\mathbf{G}_\phi, \overline{\mathbb{Q}}_\ell) \mapsto \mathrm{ind}_{W'_{F_f}}^{W'_F}(\tilde{\sigma} \otimes \rho) \in \Phi(\mathbf{G}, \overline{\mathbb{Q}}_\ell),$$

For example, denoting by Sp_e the special representation of dimension e (associated to the Steinberg representation), we see that $\xi_{\varphi,*}(\mathrm{Sp}_e) = \mathrm{Sp}_e \otimes \mathrm{Ind}_{W_{F_f}}^{W_F}(\tilde{\sigma})$. Let us translate this in terms of irreducible representations. Let π be the supercuspidal representation of $\mathrm{GL}_{f\dim(\sigma)}(F)$ that corresponds to $\mathrm{Ind}_{W_{F_f}}^{W_F}(\tilde{\sigma})$ via the LLC. To any pair (χ, a) with χ an unramified character of $F_f^\times$ and $a \in \mathbb{N}$, we associate the segment $\Delta_\pi(\chi, a) = (\pi_\chi, \pi_\chi \otimes \nu, \cdots, \pi_\chi \otimes \nu^{a-1})$ where $\nu = |\det|_F$ and $\pi_\chi = \pi \otimes (\chi^{1/f} \circ \det)$ (which is independent of the choice of an f^{th}-root of χ). This extends to a bijection $m \mapsto m_\pi$ between multisegments of unramified characters of $F_f^\times$ and "multisegments of type π". Then the formula above shows that the transfer map $\xi_{\varphi,*} : \mathrm{Irr}_1(G_\phi) \longrightarrow \mathrm{Irr}_\phi(G)$ takes $Z(m)$ to $Z(m_\pi)$ in Zelevinski's notation (or equivalently $L(m)$ to $L(m_\pi)$), compare [12, §2].

Now let us put $\mathcal{H}_\phi := \mathcal{H}(q_F^f, e)$. Thanks to their theory of simple types, Bushnell and Kutzko produce "natural" equivalence*s* of categories between $\mathrm{Mod}(\mathcal{H}_\phi)$ and $\mathrm{Rep}_\phi(\mathrm{GL}_n(F))$ [3, Thm 7.5.7]. These equivalences are unramified twists of each other [3, Prop. 7.5.10]. In light of the above, we will normalize the equivalence so that it takes the sign character of $\mathcal{H}_\phi$ to the "generalized" Steinberg representation $\mathrm{St}_e(\pi)$. Then, the compatibility of Bushnell-Kutzko equivalences with normalized parabolic induction [3, Thm. 7.6.1] and unramified twisting [3, Prop. 7.5.12] shows that it also takes the simple module $M(m)$ associated to the multisegment m to $Z(m_\pi)$. On the other hand, as recalled previously, Borel's theorem produces a "canonical" equivalence of categories between $\mathrm{Mod}(\mathcal{H}_\phi)$ and $\mathrm{Rep}_1(G_\phi)$, that takes $M(m)$ to $Z(m)$. By composition we get an equivalence between $\mathrm{Rep}_1(G_\phi)$ and $\mathrm{Rep}_\phi(\mathrm{GL}_n(F))$ that takes $Z(m)$ to $Z(m_\pi)$, as desired.

Remark.– We may ask whether an equivalence as in statement $\mathcal{E}_F(\phi', \xi)$ is unique. In view of the above discussion, this reduces to asking whether an auto-equivalence α of $\mathrm{Mod}(\mathcal{H}(q, n))$ that "preserves simple modules" (in the sense that $\alpha(M) \simeq M$ for each simple module M) is isomorphic to the identity functor. For this, one has to compute the Picard group of $\mathcal{H}(q, n)$ over its center.

2.4.10 *"Proof" of Theorem 1.2.4.* — Here we explain how Theorem 1.2.4 follows from constructions in [5]. So we assume that $K_F = I_F^{(\ell)}$ and we consider statements $\mathcal{E}_F(\phi', \xi)$ for $\overline{\mathbb{Z}}_\ell$-blocks when both ϕ' and ξ are *tame.* Recall that this means that $\phi'_{|P_F}$, resp. $\xi_{|P_F}$, is equivalent to the trivial parameter. We note that Lemma 2.4.8 remains true if we impose tameness of ϕ', ξ and ϕ in each item i), ii) or iii). This is because a unipotent factorization of a tame parameter is tame, and Shapiro bijections preserve tameness. In this tame setting we can reduce further these statements as follows.

Lemma. — Assertions i), ii) and iii) of Lemma 2.4.8 restricted to tame parameters are equivalent to :

iv) Statement $\mathcal{E}_F(\phi', \xi)$ is true in the following cases :

(a) ϕ' is tame and ξ is an unramified automorphic induction ${}^L\mathrm{Res}_{F_f|F}(\mathrm{GL}_{n/f}) \longrightarrow {}^L\mathrm{GL}_n$

(b) $\phi' = 1$ and ξ is a totally ℓ'-ramified base change ${}^L\mathrm{GL}_n \longrightarrow {}^L\mathrm{Res}_{F'|F}\mathrm{GL}_n$.

Moreover the same equivalence holds for statements $\mathcal{E}_F(\phi', \xi)^-$ and $\mathcal{U}_F(\phi)^-$.

Proof. $i) \Rightarrow iv)$ is clear, so we only need to check that $iv) \Rightarrow iii)$, and in fact it is sufficient to prove that $iv)(a)$ implies property $\mathcal{U}_F(\phi)$ for tame parameters ϕ of $\mathbf{G} = \mathrm{GL}_n$. Write $\phi = \hat{\phi} \times \mathrm{Id}_{I_F^{(\ell)}}$ with $\hat{\phi}$ an n-dimensional representation of $I_F^{(\ell)}/I_F$. By 2.4.6, we know that if $\mathbf{G}_\phi$ is not quasi-simple then any unipotent factorization factors through some Levi subgroup. Thanks to Example 2.4.5, we may thus assume that $\mathbf{G}_\phi$ is quasi-simple. In this case, 2.4.6 tells us that we can extend $\hat{\phi}$ to $\hat{\varphi} \simeq \overline{\mathbb{Q}}_\ell^e \otimes \mathrm{Ind}_{W_\sigma}^{W_F}(\tilde{\sigma})$, where σ is an irreducible summand of $\hat{\phi}$, and $\tilde{\sigma}$ is an extension of σ to its normalizer W_σ in W_F. Factorization (2.2.3) of φ then reads

$$\varphi:\ W_F \xrightarrow{(1,\mathrm{Id}_W)} {}^L\mathbf{G}_\phi = C_{\hat{\mathbf{G}}}(\phi) \rtimes_{\alpha_\varphi} W_F \xrightarrow{\mathrm{Id}\,.\varphi} \mathrm{GL}_n \times W_F,$$

with $\mathbf{G}_\phi = \mathrm{Res}_{F_\sigma|F}(\mathrm{GL}_e)$ for the finite extension F_σ such that $W_{F_\sigma} = W_\sigma$. In our tame context, since I_F/P_F is abelian, W_σ contains I_F hence $F_\sigma = F_f$ is the unramified extension of some degree f over F. Moreover, since σ has dimension 1, we have $f = n/e$ and $C_{\hat{\mathbf{G}}}(\phi)$ identifies with the diagonal Levi subgroup $(\mathrm{GL}_e)^f$ of GL_n.

Now, looking at the unipotent factorization above, we see that it involves the "right groups" $\mathrm{Res}_{F_f|F}(\mathrm{GL}_e)$ and GL_n, but the morphism of L-groups $\mathrm{Id}\,.\varphi$ is *not* the automorphic induction morphism. So we look for another factorization (not unipotent) of φ, involving the same groups but with the automorphic induction morphism.

We know that, as with any extension of ϕ, the group $\hat{\varphi}(W_F)$ is contained in the normalizer $\mathcal{N} = \mathcal{N}_{\hat{\mathbf{G}}}(C_{\hat{\mathbf{G}}}(\phi))$. For formal reasons, $\varphi(W_F)$ is therefore contained in the subgroup $\mathcal{N} \times_{\mathcal{N}/\mathcal{N}^\circ} W_F$ of ${}^L\mathbf{G}$, where the fibered product is for the composition $W_F \xrightarrow{\hat{\varphi}} \mathcal{N} \longrightarrow \mathcal{N}/\mathcal{N}^\circ$. Actually, if ε denotes any diagonal épinglage of $C_{\hat{\mathbf{G}}}(\phi) \simeq (\mathrm{GL}_e)^f$, we know that $\hat{\varphi}(W_F)$ is contained in the normalizer $\mathcal{N}_\varepsilon$ of this épinglage in $\mathcal{N}$, and therefore $\varphi(W_F) \subset \mathcal{N}_\varepsilon \times_{\mathcal{N}/\mathcal{N}^\circ} W_F$.

Now the point is that $\mathcal{N}^\circ = C_{\hat{\mathbf{G}}}(\phi)$ (because it is a Levi subgroup), $\mathcal{N}_\varepsilon^\circ = Z(C_{\hat{\mathbf{G}}}(\phi))$, and we can find a section morphism $\mathcal{N}_\varepsilon/\mathcal{N}_\varepsilon^\circ = \mathcal{N}/\mathcal{N}^\circ \hookrightarrow \mathcal{N}_\varepsilon$ (taking permutation matrices). Then the action α_ε of W_F on $C_{\hat{\mathbf{G}}}(\phi)$ through $\mathcal{N}/\mathcal{N}^\circ$ makes the semi-direct product $C_{\hat{\mathbf{G}}}(\phi) \rtimes_{\alpha_\varepsilon} W_F$ isomorphic to ${}^L\mathbf{G}_\phi$, and identifies the semi-direct product $Z(C_{\hat{\mathbf{G}}}(\phi)) \rtimes_{\alpha_\varepsilon} W_F$ with $Z(\hat{\mathbf{G}}_\phi) \rtimes W_F \subset {}^L\mathbf{G}_\phi$. Moreover this section induces an isomorphism $C_{\hat{\mathbf{G}}}(\phi) \rtimes (\mathcal{N}/\mathcal{N}^\circ) \xrightarrow{\sim} \mathcal{N}$, which in turn induces an isomorphism

$$\iota:\ {}^L\mathbf{G}_\phi = C_{\hat{\mathbf{G}}}(\phi) \rtimes_\alpha W_F = (C_{\hat{\mathbf{G}}}(\phi) \rtimes (\mathcal{N}/\mathcal{N}^\circ)) \times_{\mathcal{N}/\mathcal{N}^\circ} W_F \xrightarrow{\sim} \mathcal{N} \times_{\mathcal{N}/\mathcal{N}^\circ} W_F.$$

Now consider $\varphi' := \iota^{-1} \circ \varphi : W_F \longrightarrow {}^L\mathbf{G}_\phi$ and ξ the composition of ι with the inclusion of $\mathcal{N} \times_{\mathcal{N}/\mathcal{N}^\circ} W_F$ into ${}^L\mathbf{G}$. By construction, we have a factorization

$$\varphi:\ W_F \xrightarrow{\varphi'} {}^L\mathbf{G}_\phi = {}^L\mathrm{Res}_{F_f|F}\mathrm{GL}_e \xrightarrow{\xi} {}^L\mathbf{G}$$

and ξ is the automorphic induction L-morphism. Restricting to $I_F^{(\ell)}$, we get a factorization

$$\phi:\ I_F^{(\ell)} \xrightarrow{\phi'} {}^L\mathbf{G}_\phi = {}^L\mathrm{Res}_{F_f|F}\mathrm{GL}_e \xrightarrow{\xi} {}^L\mathbf{G}$$

where ϕ' is extendable to W_F (namely to φ') and ξ induces an isomorphism on centralizers.

Now, hypothesis iv)(a) provides us with an equivalence of categories $\mathrm{Rep}_{\phi'}(\mathbf{G}_\phi) \xrightarrow{\sim} \mathrm{Rep}_\phi(\mathbf{G})$, but what we need is an equivalence $\mathrm{Rep}_1(\mathbf{G}_\phi) \xrightarrow{\sim} \mathrm{Rep}_\phi(\mathbf{G})$. For this, observe that φ' factors through $Z(\hat{\mathbf{G}}_\phi) \rtimes W_F$ (because $\hat{\varphi}(W_F) \subset \mathcal{N}_\varepsilon$). Therefore φ' corresponds to a character $G_\phi \xrightarrow{\chi'} \overline{\mathbb{Q}}_\ell^\times$ as in [2, 10.2]. Twisting by this character provides an equivalence $\mathrm{Rep}_1(\mathbf{G}_\phi) \xrightarrow{\sim} \mathrm{Rep}_{\phi'}(\mathbf{G}_\phi)$ and composing with the latter gives the desired correspondence. $\square$

We close this paragraph by stating that the weakened forms of iv)(a) and iv)(b) (i.e. whithout compatibility with transfer) are proved in [5]. What is missing at the moment to get the strong form is the compatibility of the construction in loc. cit. with parabolic induction.

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